This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

PLANE AND SPHERICAL
TRIGONOMETRY.

By the same Author.

I.

PLANE AND SPHERICAL TRIGONOMETRY.

PART I., containing Rules, Examples, and Problems. This volume contains the principal Rules in Logarithms, and in Plane and Spherical Trigonometry. The Examples and Problems exceed 400, and the answer or result is given under each. Sixth edition, 4*s*. cloth.

II.

PLANE AND SPHERICAL TRIGONOMETRY.

PART II., containing the principal Formulæ, Proofs of Rules in Logarithms and Trigonometry, &c. given in PART I.; and a series of upwards of 300 Analytical Examples and Exercises, with Hints and Answers. Sixth edition, 4*s*. cloth.

The two Parts complete in one volume, 7*s*. 6*d*. cloth.

III.

PROBLEMS IN ASTRONOMY, SURVEYING, and NAVIGATION, with their SOLUTIONS. Designed for facilitating Self-instruction in the principles of Navigation and Nautical Astronomy. This volume contains the SOLUTIONS of all the Problems given in Trigonometry, Part I. Second edition, 6*s*. cloth.

IV.

HAND-BOOK FOR THE STARS;

or Rules for finding the Names and Positions of all the Stars of the First and Second Magnitude. Third edition, 4*s*. 6*d*. cloth.

V.

NAVIGATION AND NAUTICAL ASTRONOMY.

PART I., containing Rules and Examples. 7*s*. 6*d*. cloth.

VI.

NAVIGATION AND NAUTICAL ASTRONOMY.

PART II., containing Investigations and Proofs of the principal Rules and Corrections. 7*s*. 6*d*. cloth.

The two Parts complete in one volume. 14*s*. cloth.

LONDON: LONGMANS AND CO., PATERNOSTER ROW.

PLANE AND SPHERICAL
TRIGONOMETRY.

PART I.

CONTAINING RULES, EXAMPLES, AND PROBLEMS.

BY

H. W. JEANS, F.R.A.S.

LATE MATHEMATICAL MASTER
AND EXAMINER AT THE ROYAL NAVAL COLLEGE, PORTSMOUTH;
FORMERLY MATHEMATICAL MASTER IN THE ROYAL MILITARY ACADEMY, WOOLWICH;
AND AN EXAMINER OF MERCHANT OFFICERS IN NAUTICAL ASTRONOMY
UNDER THE TRINITY BOARD.

DESIGNED PRINCIPALLY FOR NAVAL STUDENTS AND ENGINEERS.

BIBLIOTHECA
JUN 15 73
BODLEIANA

Sixth Edition.

Rec? 18 Feb. 73.

LONDON:
LONGMANS, GREEN, & CO., PATERNOSTER ROW.
1873.

183 f. 32

LONDON:
ROBSON AND SONS, PRINTERS, PANCRAS ROAD, N.W.

PREFACE.

THE following pages contain the Practical Rules in Plane and Spherical Trigonometry. The investigations of these Rules, requiring some knowledge of Mathematics, are given in the Second Part of this treatise. The present volume is made to consist entirely of the Rules and their applications, and the demonstrations are placed in Part II., in order that the student may be enabled to proceed with Problems in Surveying, Navigation, and Astronomy, as soon as he is acquainted with vulgar and decimal fractions, and has acquired a sufficient knowledge of algebra to work an easy equation.

The Problems at the end of the book have been selected chiefly for Naval Students; most of them can be solved by means of the Rules contained in the book. The Problems marked with the letter (*a*) have been added for the use of those who have already made some progress in Mathematics: these cannot be solved by a simple application of the Rules; they will present, however, little difficulty to the student who has made himself acquainted with the contents of Part II.

The table of *log. haversines** contained in Inman's Nautical Tables, and now generally known to Naval Students, reduces considerably the labour of working out some of the Problems in Navigation: it may also be applied with equal advantage to the principal cases in Plane and Spherical Trigonometry. Rules have, accordingly, been now for the first time adapted to this table.* Other rules are also given for the same cases suited to the Logarithmic Tables in more general use, such as Hutton's, or those in Norie's or Riddle's works on Navigation.

---

* This table, under the name of logarithmic versed sines, may be found in Mendoza Rios, calculated to *five* places of decimals; the table in Norie, called *log. rising*, may be formed from it by the addition of the constant log. $\bar{5}\cdot30103$. The above table of Mendoza Rios was *re-calculated by the Author* of the present work, and carried to

The young student who may use this volume as an Introduction to Navigation will not find it necessary to read, at first, more than a certain portion of it. The Articles marked in the table of contents with the letter (n) may be sufficient for this purpose. The Examples under each of the Rules thus selected should be worked out, with the exception of a few of the more difficult. The Examples that may be omitted at the first reading are separated from the others by a line. Such student may also solve a few of the Problems at the end of the book, such as [1] to [20]; [39] to [47]; and, as

---

six places of decimals, and arranged in a more convenient form for use; it was first inserted in Dr. Inman's Nautical Tables, in 1837, under the name of the table of *log. haversines.* Subsequently a similar table was printed in a work on Navigation by Lieut. Raper, and called by him *log. sine square.* The reason for adopting the two last-mentioned names will appear from considering the formula by means of which the table was constructed from the common table of log. sines, namely:

$$\text{sin.}^2\frac{A}{2}=\tfrac{1}{2}\text{ vers. }A\ \text{[Trig. Part II. formula (29)].}$$

or, log. sin. square$\frac{A}{2}$=log. half versed sine A.

Raper derived the name of his table from the first side of this equation; Inman from the last, contracting half versed sine into haversine.

* In the later editions of Inman's Tables a table of *half* log. haversines has been added. This new table will be found to diminish the mechanical labour of working out many of the principal problems *in Nautical Astronomy* and Trigonometry.

an INTRODUCTION TO NAUTICAL ASTRONOMY, Problems [102] to [113]; [135], [136], [143], [144]. The remainder of the book may be omitted until he has read Part II.

The answers to the Examples and Problems have generally been taken from the tables *by inspection;* that is, if Inman's tables were at hand, to the nearest 15″; if any others, to the nearest minute or half minute. It is seldom necessary in practical questions of this kind to proportion to the nearest second; and in Nautical Problems the above degree of accuracy will in almost every case be sufficient.

Those who have not the assistance of a tutor, and who have therefore to rely on their own exertions for the knowledge they may require, will find the volume of *Problems in Astronomy, Surveying, and Navigation, with their Solutions,* published separately, and which in fact is the Key to the problems at the end of this book, of great assistance.

H. W. J.

Langstone, near Havant,
Dec. 1872.

CONTENTS.

PLANE TRIGONOMETRY.

The articles marked (*o*) may be omitted at first by the student. Those marked (*n*) form an introduction to Navigation and Nautical Astronomy.

CHAPTER I.

PAGE
(*n*) Trigonometrical ratios, viz. sine, tangent, &c. 2
(*n*) To find the index of the logarithm of any number. . . . 4
(*n*) To find the logarithm of any number 6
(*n*) To find the natural number of a logarithm 7
(*n*) Multiplication by logarithms 9
(*n*) Division by logarithms 10
(*n*) Involution by logarithms 11
(*n*) Evolution by logarithms. 13
(*n*) To find the value of a number with a fractional index . . 14
(*n*) Proportion by logarithms 15
(*n*) To reduce algebraical formulæ to logarithms. 16
(*o*) Exponential equations, &c. 21
(*o*) To find the value of an expression, when the index of the logarithm is negative. 22
(*n*) To take quantities out of table of logarithmic sines, &c. . 23
(*n*) When the given angle is greater than 90° 24
(*n*) To take quantities out of table of natural versines. . . . 25
(*n*) To reduce trigonometrical formulæ to logarithms . . . 25
(*n*) To reduce logarithmic formulæ to tabular logarithms . . 26
(*o*) To calculate a natural versine from the table of logarithmic sines, &c. 28
(*o*) Use of algebraical signs + and − to determine the magnitude of angles 29
(*o*) To determine the algebraic sign of a trigonometrical ratio in a formula where the other terms are known 30
(*o*) To determine the numerical value of trigonometrical expressions connected by the signs + and − 33

APPLICATION OF LOGARITHMS TO ARITHMETIC, MENSURATION, &c.; VIZ.

PAGE
(*o*) Examples in progression and astronomy 34
(*o*) Formulæ for the mensuration of the principal regular bodies. 36
(*o*) Examples in mensuration 37

CHAPTER II.

RULES IN PLANE TRIGONOMETRY.

(*n*) Rule I. Three sides of a plane triangle being given, to find an angle (using table of haversines). First method . . 44
(*n*) Three sides of a plane triangle being given, to find an angle (without using haversines). Second method . . . 45
(*n*) Rule II. Of two sides and two opposite angles, any three being given, to find the fourth 47
(*o*) Ambiguous case 50
(*n*) Rule III. Two sides and included angle being given, to find the remaining angles 52
(*n*) Rule IV. Two sides and included angle being given, to find the remaining side (using haversines). First method . 54
(*n*) Two sides and included angle being given, to find the remaining side (without haversines). Second method . . . 55
(*n*) Rule V. Right-angled plane triangles 57
(*n*) Rule VI. Two sides and included angle being given, to find the area 60
(*n*) Rule VII. Three sides of a plane triangle being given, to find the area 60

SPHERICAL TRIGONOMETRY.

CHAPTER III.

RULES IN SPHERICAL TRIGONOMETRY.

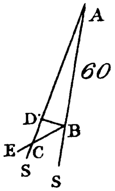
(*n*) Rule VIII. Three sides of a spherical triangle being given, to find an angle (using haversines). First method . . . 62
(*n*) Three sides of a spherical triangle being given, to find an angle (without haversines). Second method . . . 62
(*n*) Rule IX. Two sides and included angle being given, to find the third side (using haversines). First method . . 64

PAGE
(*n*) Two sides and included angle being given, to find the third side (without haversines). Second method . . . 65
(*n*) Two sides and included angle being given, to find the third side (using table of sines, &c.). Third method . . . 65
(*n*) Rule X. Of two sides and two opposite angles, any three being given, to find the fourth 68
(*n*) Rule XI. Two sides and included angle being given, to find the other two angles 69
(*n*) Rule XII. Two angles and included side being given, to find the other two sides 70
(*n*) Rule XIII. Rules for right-angled spherical triangles . . 72
(*n*) Rule XIV. Rules for quadrantal triangles 75

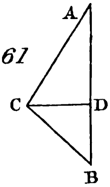
(*n*) EXAMINATION PAPERS IN PRECEDING RULES . . . 78

CHAPTER IV.

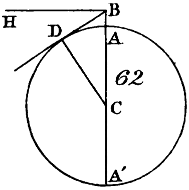
PROBLEMS IN SURVEYING, &c. 82
PROBLEMS IN SURVEYING, ASTRONOMY, AND NAVIGATION . 106
DEFINITIONS IN ASTRONOMY 123
DEFINITIONS IN NAVIGATION 128
EXAMPLES AND PROBLEMS IN NAUTICAL ASTRONOMY AND NAVIGATION 129



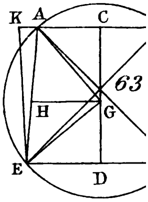
60



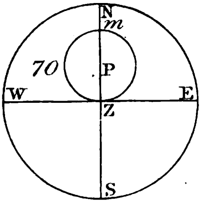
61



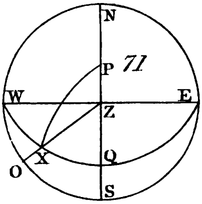
62



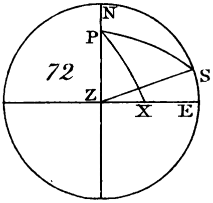
63



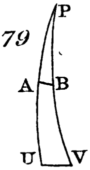
70



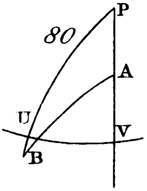
71



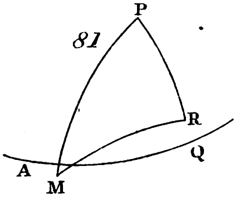
72



79

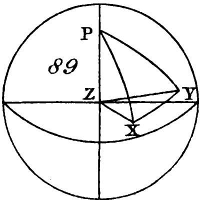


80

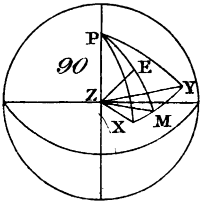


81

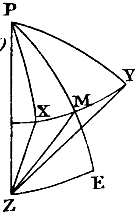
82



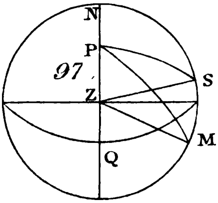
89



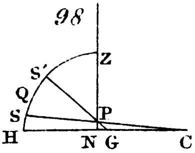
90



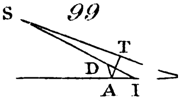
90



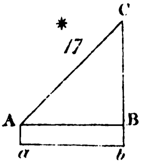
97

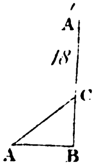


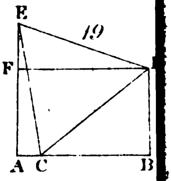
98

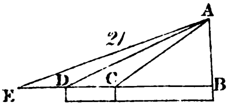


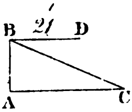
99

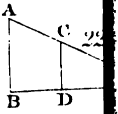

*
17
C
A
B
a
b

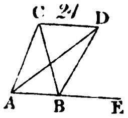

A
18
C
A
B

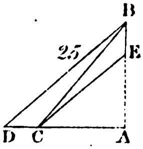

E
19
F
A
C
B

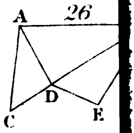

21
A
E
D
C
B

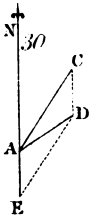

B
21
D
A
C

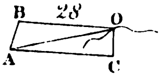

A
C
22
B
D

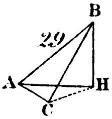

C
24
D
A
B
E

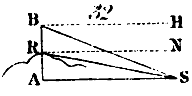

B
25
E
D
C
A

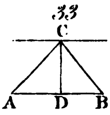

A
26
D
C
E

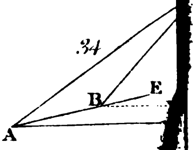

N
30
C
D
A
E


B
28
O
A
C


B
29
A
H
C


32
B
H
R
N
A
S


33
C
A
D
B


34
B
E
A

PLANE TRIGONOMETRY.

PART I.

CHAPTER I.

(1). ONE of the principal uses of Trigonometry is to furnish rules for finding any *part* of a triangle (that is, an angle or a side), when the number of parts already known is sufficient; and this is the case, in general, when three of the parts of the triangle are known.

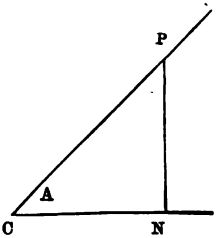
(2). Considered as a branch of mathematics, Trigonometry teaches us the method of investigating certain relations of angles denominated *formulæ.* Trigonometry is therefore usually defined to be, that branch of mathematics which treats of the relations which the sides and angles of triangles have to one another, and of the general relations of angles.

(3). The First Part of this treatise contains the Rules for solving Plane and Spherical Triangles, with a large collection of examples and problems for practice. The problems have been selected principally to serve as an Introduction to Navigation and Nautical Astronomy.

(4). In the Second Part will be investigated the fundamental formulæ in Plane and Spherical Trigonometry. These formulæ continually recur in mathematics, and the student whose views extend beyond the practical part of Navigation must make himself well acquainted with them. In the Second Part also, the rules given in the First Part are investigated and proved.

(5). At present it will be sufficient to name only a few of the terms used in Trigonometry; the principal of which are the fractions called the *Trigonometrical ratios.*

Definitions of the Trigonometrical ratios.



Let the angle PCN be denoted by A; and let PN be drawn from any point P perpendicular to CN, thus forming a right-angled triangle: then the *trigonometrical ratios* are the fractions that can be formed of the three sides of the right-angled triangle PCN, taking them two and two together. It is evident that only six such fractions can be formed in this manner, namely, $\frac{PN}{CP}$, $\frac{CP}{PN}$, $\frac{PN}{CN}$, $\frac{CN}{PN}$, $\frac{CP}{CN}$, and $\frac{CN}{CP}$.* To connect these fractions with one of the angles of the triangle, as A, let us call the side CP opposite the right angle the hypothenuse, the side PN opposite to the angle in question the perpendicular, and the remaining side CN, adjacent to the angle, the base; then the trigonometrical ratios of an angle are named as follows: the perpendicular divided by the hypothenuse is called the SINE of the angle; the hypothenuse divided by the perpendicular is called the COSECANT of the angle; the perpendicular divided by the base is called the TANGENT of the angle; the base divided by the perpendicular is called the COTANGENT of the angle; the hypothenuse divided by the base is called the SECANT of the angle; and the base divided by the hypothenuse is called the COSINE of the angle. Another term used in Trigonometry is the VERSINE of an angle: this quantity is equivalent to the excess of unity over the cosine. These definitions will be more readily learned by writing them down in the form of fractions, thus:

---

* The use and importance of the trigonometrical ratios will be more clearly seen in Part II. We may here briefly remark, that since we are not able to compare together the angles and sides of a triangle (for we cannot say that an angle is three or more times greater than a side, since these quantities have different units of measurement), it has been found convenient to take certain lines or fractions, such as $\frac{PN}{CP}$, &c., and compute their numerical values for different angles, and record these values in tables. Now these fractions will evidently depend for their value on the magnitude of the angles themselves; and if we know the fraction, we shall know, by means of the table, the angle corresponding to it. Moreover, we can compare the side of a triangle with this fraction, since both quantities are expressed in the same kind of measure, namely, linear measure; and then, by reference to the table, discover the relation of the side to the angle itself.

$\frac{\mathrm{PN}}{\mathrm{CP}}$ or $\frac{\text{perpendicular}}{\text{hypothenuse}}$ is called the *sine* of the angle A (*a*)
$\frac{\mathrm{CP}}{\mathrm{PN}}$ or $\frac{\text{hyp.}}{\text{perp.}}$ . . . *cosecant* . . . (*b*)
$\frac{\mathrm{PN}}{\mathrm{CN}}$ or $\frac{\text{perp.}}{\text{base}}$ . . . *tangent* . . . (*c*)
$\frac{\mathrm{CN}}{\mathrm{PN}}$ or $\frac{\text{base}}{\text{perp.}}$ . . . *cotangent* . . (*d*)
$\frac{\mathrm{CP}}{\mathrm{CN}}$ or $\frac{\text{hyp.}}{\text{base}}$ . . . *secant* . . . (*e*)
$\frac{\mathrm{CN}}{\mathrm{CP}}$ or $\frac{\text{base}}{\text{hyp.}}$ . . . *cosine* . . . (*f*)
Also, 1 — cosine of A is called the *versine* of angle A.

The values of these ratios or fractions being calculated for every minute or quarter of a minute from 0° to 45°, constitute the tables of sines, tangents, &c., in general use: by means of which, and certain other tables of numbers called *logarithms*, problems in Trigonometry and Astronomy are usually solved.

(6). Before, therefore, we proceed to apply Rules for solving trigonometrical problems, we must, in order to familiarise the student with logarithmic computation, show the extensive use of logarithms in shortening and simplifying complex and tedious arithmetical calculations; in fact, we shall find that, by their means, processes of multiplication are reduced to addition, division to subtraction, the formation of powers to multiplication, and the extraction of roots to division, &c.

(7). In Part II. it is shown what is meant by a logarithm, and how the logarithms of numbers have been computed and arranged in tables, so that now the logarithm of any given number may be found by inspection; and conversely, if any logarithm is given, the number corresponding to it (which, for the sake of distinction, is called the natural number) may also be found.

LOGARITHMS.*

EXAMPLES AND EXERCISES IN THE USE OF LOGARITHMS.

Logarithms usually consist of a whole number and a decimal fraction: the whole number may be positive or negative; the decimal part of the logarithm is always *positive:* thus, the student will find by referring to a table of logarithms that the logarithm of the natural number 200 is 2·301030; the whole number, 2, is called the *index* or characteristic; the decimal part, namely ·301030, is called the *mantissa.* The mantissa is always positive; the index may be positive or negative. Thus the logarithm 2·301030 is + 2 and + ·301030; but the logarithm $\bar{2}$·301030 is in fact − 2 and + 0·301030. In the tables the decimal part or mantissa only is inserted; the index being found as follows:

RULE I.

(8). *To find the index of the logarithm of a whole or mixed number.*

The index of the logarithm is less by one than the number of *integral* places contained in the number.

Thus the *index* of the log. of	24	is	1
of	2463	is	3
of	147·2	is	2
of	14·72	is	1
of	1·472	is	0

---

* The definitions and analytical proofs of the rules in logarithms are given in the Second Part of this book.

RULE II.

(9). *To find the index of the logarithm of a decimal fraction.*

Consider the decimal fraction as a whole number, and find the index of its logarithm by the preceding rule: then subtract the number thus found from the number of figures in the given decimal; the remainder, with the negative sign (—) placed over it, is the index required.

Thus, the index of the logarithm of ·00452 is $\bar{3}$; for the index of the logarithm of 452 is 2, by Rule I., and the number of figures in ·00452 is 5: hence $2-5=-3$, or as it must be written $\bar{3}$, to show that the mantissa or decimal part of the logarithm is still positive.*

EXAMPLES.

The index of log. ·123 is . . . $\bar{1}$
of log. ·0123 is . . . $\bar{2}$
of log. ·00064 is . . . $\bar{4}$
of log. ·000000721 is . . . $\bar{7}$

(10). *To find the index of the logarithm of a vulgar fraction.*

Reduce the fraction to an equivalent decimal fraction; the index of its logarithm may then be found by Rule II.

Thus, the index of log. $\frac{1}{8}$ or of log. ·125 is . . . . $\bar{1}$
of log. $24\frac{2}{5}$ or of log. 24·4 is . . . 1
of log. $\frac{1}{25}$ or of log. ·04 is . . . . $\bar{2}$
of log. $\frac{1}{120}$ or of log. ·00833 is . . $\bar{3}$

---

* Hence, to reduce a logarithm with a negative index to a quantity wholly negative, and the converse, we must proceed thus: $\bar{3}\cdot602060$, or its equivalent $-3+\cdot602060=-2\cdot397940$; therefore, also, $-4\cdot301415=\bar{5}\cdot698585$.

RULE III.

(11). *To take out the logarithm of any given number from the table.*

The rule for taking out a logarithm for any given number will be found in the explanations which accompany the table, to which the student is referred.* We will here insert a collection of numbers for exercises whose logarithms it is required to find from the tables.

EXAMPLES.

Let it be required to take out from the tables the logarithms of 2482, of 24·82, and of ·002482.

Looking for 2482 in the table, we see opposite to it ·394802, the decimal part of its logarithm; and this part, by the explanation accompanying the table, is the same for each of the above numbers: the index or whole number to be prefixed, found by Rules I. or XI., will indicate the number of integral figures in the natural number.

Thus the log. of 2482 is . . . . 3·394802
log. of 24·82 is . . . . 1·394802
log. of ·002482 is . . . . $\bar{3}$·394802

Find from the tables the logarithms of the following natural numbers:

Nat. No.			
24	*Answer.*	Its logarithm is	1·380211
248	,,	. . . .	2·394452
2480	,,	. . . .	3·394452
1476	,,	. . . .	3·169086
14·06	,,	. . . .	1·147985
1·406	,,	. . . .	0·147985
3847	,,	. . . .	3·585122

---

* It might mislead the student to give in this place a rule for taking out a logarithm, as it must be made to depend on the peculiar arrangement of the table in his hands: nor is it necessary, since ample directions for finding the logarithms of numbers are always appended to the table, either at the beginning or end.

Nat. No.			
38475*	*Ans.*	Its logarithm is	4·585178
384·75	„	. . . .	2·585178
384·757	„	. . . .	2·585186
4196·534	„	. . . .	3·622890
121004·15	„	. . . .	5·082800
$48\frac{3}{4}$ or 48·75	„	. . . .	1·687975
$49\frac{1}{4}$ or 49·25	„	. . . .	1·692406
2000	„	. . . .	3·301030
200000	„	. . . .	5·301030
$1000\frac{1}{2}$	„	. . . .	3·000216
2·4	„	. . . .	0·380211
·24	„	. . . .	$\bar{1}$·380211
·024	„	. . . .	$\bar{2}$·380211
·000035	„	. . . .	$\bar{5}$·544068
1415216	„	. . . .	6·150822
$\frac{3}{4}$ or ·75	„	. . . .	$\bar{1}$·875061
$\frac{1}{2}$	„	. . . .	$\bar{1}$·698970
$\frac{1}{25}$	„	. . . .	$\bar{2}$·602060
$\frac{1}{2000}$	„	. . . .	$\bar{4}$·698970
·004	„	. . . .	$\bar{3}$·602060
·04	„	. . . .	$\bar{2}$·602060
·6945	„	. . . .	$\bar{1}$·841672
·75234	„	. . . .	$\bar{1}$·876414
$90\frac{1}{25}$	„	. . . .	1·954435
$1000\frac{1}{125}$	„	. . . .	3·000003
$387\frac{1}{2}$	„	. . . .	2·588272
$726\frac{1}{4}$	„	. . . .	2·861086

RULE IV.

(12). *To take out the natural number corresponding to any given logarithm.*

For the method of taking out the figures that constitute the natural number, see Explanation accompanying the tables.

---

* When the natural number (as in this example) consists of more than four figures (the extent of logarithmic tables in general use), the fifth and all subsequent figures can be found by the rules given with the *tables (see Explanation* of tables).

To find where the decimal point in the natural number taken out must be placed.

First. When the index of the given logarithm is positive, to find its natural number.

The number of integral places in the number taken out will be *one* more than the index. Thus, if the index of a logarithm is 2, the natural number contains three integral places; and so on: the rest of the figures in the number taken out are decimals.

EXAMPLES.

Given the logarithm 2·477121, to find its natural number.

Entering the table with the decimal part ·477121, we find the number corresponding to it to be 3, or 30, or 300, or 3000, &c.; but, as the index of the logarithm is 2, the natural number must contain three integral figures. Hence the natural number of 2·477121 is 300.

Required the natural numbers of the following logarithms:

Logarithms.			
1·380211	*Ans.*	Nat. No. is	24
2·394452	,,	. . . .	248
3·394452	,,	. . . .	2480
6·394452	,,	. . . .	2480000
2·415671	,,	. . . .	260·417
1·415674	,,	. . . .	26·042
2·310101	,,	. . . .	204·221
4·196171	,,	. . . .	15709·83
0·217845	,,	. . . .	1·65137
1·841569	,,	. . . .	69·4335
2·841989	,,	. . . .	695·0064
5·082800	,,	. . . .	121004·19
0·147985	,,	. . . .	1·406
0·394452	,,	. . . .	2·48

(13). *Second.* When the index of the given logarithm is negative, to find its natural number.

The natural number in this case will be a decimal. To find the number of ciphers (if any) to be prefixed to the figures taken from the tables, diminish the index by 1 (without regarding the negative sign); the remainder will denote the number of ciphers to be prefixed to the figures of the number taken from the tables: thus, if the index is $\bar{4}$, prefix three ciphers; if $\bar{2}$, prefix one cipher; if $\bar{1}$, the decimal mark is placed next to the figures taken out; and so on.

EXAMPLES.

Given the logarithm $\bar{4}\cdot394452$, to find its natural number.

Entering the table with the decimal part ·394452, we see the natural number opposite to it is 248: to this number prefix three ciphers, since the index of the logarithm is $\bar{4}$; hence the natural number of $\bar{4}\cdot394452$ is ·000248.

Required the natural numbers of the following logarithms:

Logarithms.			
$\bar{2}\cdot380211$	*Ans.*	Nat. No. is	·024
$\bar{5}\cdot544068$	"	. . . .	·000035
$\bar{1}\cdot841672$	"	. . . .	·6945
$\bar{7}\cdot875061$	"	. . . .	·00000075
$\bar{3}\cdot602060$	"	. . . .	·004
$\bar{2}\cdot394452$	"	. . . .	·0248

RULE V.

Multiplication by logarithms.

(14). Take out the logarithms of the given numbers from the tables; add them together: their sum will be the logarithm of the product of the given numbers; the natural number corresponding to which is therefore the product required.*

---

* In Part II. it is shown that log. $nmp\ldots\ldots = \log.n + \log.m + \log.p + \ldots\ldots$

EXAMPLES.

Required the product of the following quantities :

[1].	47 × 1·405 × 84	*Ans.*	5546·94
[2].	·47 × 140·5 × ·0084	,,	·5547

Calculation.

[1].		[2].	
log. 47 . .	1·672098	log. ·47 . .	$\bar{1}$·672098
,, 1·405 .	0·147676	,, 140·5 . .	2·147676
,, 84 . .	1·924279	,, ·0084 .	$\bar{3}$·924279
log. product .	3·744053	log. product . .	$\bar{1}$·744053
∴ product	5546·94	∴ product .	·5547

Required the product of the following quantities:

1.	72 × 96 × 124 × ·05	*Ans.*	42854
2.	84 × 96	,,	8064
3.	6 × 4 × 12 × 32	,,	9216
4.	64 × 362 × ·4	,,	9267
5.	36 × 48 × 62 × ·4	,,	42854
6.	1234 × 9671 × ·00617	,,	73632
7.	2·4 × ·007 × ·54 × ·1	,,	·0009072
8.	784 × ·000079 × ·0000036	,,	·0000002229

RULE VI.

Division by logarithms.

(15). From the logarithm of the dividend subtract the logarithm of the divisor: the remainder will be the logarithm of the quotient; the natural number corresponding to which will be the quotient required.*

EXAMPLES.

[1].	Divide 472 by 32·2	*Ans.*	14·66
[2].	. . ·0472 by 3·22	,,	·01466

---

* For (by Part II.) log. $\frac{n}{m}$=log. n—log. m.

Calculation.

[1].		[2].	
log. 472 . .	2·673942	log. ·0472 . .	$\bar{2}$·673942
log. 32·2 . .	1·507856	log. 3·22 . .	0·507856
log. quotient .	1·166086	log. quotient .	$\bar{2}$·166086
∴ quotient .	14·66	∴ quotient .	·01466

Find the value of the following expressions:

9.	68÷34	*Ans.* 2
10.	96÷1·6	„ 60
11.	2004·64÷34	„ 58·96
12.	19÷72	„ 0·26389
13.	19÷·72	„ 26·389
14.	$\frac{242 \times 559 \times 63}{781 \times 432}$	„ 25·26
15.	$\frac{84 \times \cdot00769 \times \cdot683}{598 \times \cdot0000146 \times \cdot039}$	„ 1295·71
16.	1÷3·2 or $\frac{1}{3\cdot2}$	„ ·3125
17.	1÷·45 or $\frac{1}{\cdot45}$	„ 2·222
18.	1÷·0004572 or $\frac{1}{\cdot0004572}$	„ 2187·23

Rule VII.

Involution, or raising of powers by logarithms.

(16). *First.* When the quantity to be raised to a power is a whole or mixed number.

Multiply the logarithm of the quantity to be raised by the number denoting the power; and the product will be the logarithm of the power, the natural number of which is the power required.*

EXAMPLES.

Required the 16th power of 1·05, or the value of $(1\cdot05)^{16}$.

---

* *For log.* $(n)^r = r \log. n$ (see Part II.).

Calculation.

$$\begin{array}{lr} \log.\ 1{\cdot}05\ .\ . & 0{\cdot}021189 \\ & 16 \\ \hline & 127134 \\ & 21189 \\ \hline \log.\ (1{\cdot}05)^{16}\ . & {\cdot}339024 \end{array}$$

$$\therefore (1{\cdot}05)^{16} = 2{\cdot}18285$$

19. Required the 6th power of 4·7215 *Ans.* 11078
20. . . . 3d . . $12\frac{1}{2}$ „ 1953·127
21. . . . 150th . . 1·05 „ 1507·82
22. . . . 200th . . 1·0125 „ 11·989
23. . . . 4th . . $7\frac{4}{5}$ „ 3701·53
24. . . 1000th . . 1·0125 „ 247742·3

(17). *Second.* When the quantity to be raised to a power is a decimal fraction.

Take from the tables the logarithm of the decimal fraction: multiply, separately, the decimal part of the logarithm so taken out, and the index of the logarithm, by the number denoting the power.

From the latter product subtract (*i.e.* add algebraically) the whole number in the former, placing the negative sign over the remainder; and then affix to it the decimal part of the former product, the natural number of which will be the required power: which find from the tables.

EXAMPLES.

Required the 10th power of ·2, or the value of $({\cdot}2)^{10}$.

$$\begin{array}{lrr} \log.\ {\cdot}2 = \bar{1}{\cdot}301030 = & \bar{1} & + {\cdot}301030 \\ & 10 & 10 \\ \hline & \overline{10} & 3{\cdot}010300 \\ & & \overline{10} \\ \hline \log.\ ({\cdot}2)^{10} & & \bar{7}{\cdot}010300 \end{array}$$

$$\therefore ({\cdot}2)^{10} = {\cdot}0000001024$$

Required the values of the following expressions:

25.	The 5th power of ·2 or $(\cdot 2)^5$	*Ans.*	·00032
26.	$(\cdot 8)^3$	„	·512
27.	$(\cdot 09163)^4$	„	·000070494
28.	$(\cdot 975)^{200}$	„	·0063241

RULE VIII.

Evolution, or extracting of roots by logarithms.

(18). *First.* When the index of the logarithm of the given number is positive; or if negative, divisible by the number denoting the root.

Take out the logarithm of the given number, and divide it by the figure denoting the root to be extracted; and the result will be the logarithm of the root, which find from the tables.*

EXAMPLES.

[1]. Required the cube root of 1234, or $\sqrt[3]{1234}$.

[2]. Required the 5th root of ·00005214, or $\sqrt[5]{\cdot 00005214}$.

Calculation.

[1].		[2].	
log. 1234 ..	$3)\underline{3\cdot 091315}$	log. ·00005214 ..	$5)\underline{\bar{5}\cdot 717171}$
log. $\sqrt[3]{1234}$ ..	1·030438	log. $\sqrt[5]{\cdot 00005214}$ ..	$\bar{1}$·143434
$\therefore \sqrt[3]{1234}$ ..	10·73	$\therefore \sqrt[5]{\cdot 00005214}$ ..	·1391

29.	Required the 5th root of	784	*Ans.*	3·79195
30.	. . . square root of	365	„	19·10498
31.	. . . cube root of	12345	„	23·11162
32.	. . . 10th root of	2	„	1·071776
33.	. . . square root of	·093	„	·304959
34.	. . . 5th root of	7·0825	„	1·479235
35.	. . . 365th root of	1·045	„	1·000121
36.	. . . cube root of	·00125	„	·1077

(19). *Second.* When the index of the logarithm of the number is *negative*, and is not divisible by the figure denoting the root.

---

* *For (Part II.)* log. $\sqrt[r]{n} = \frac{\log.\, n}{r}$.

Take out the logarithm of the number whose root is required; increase the index by the least number of units that will render it divisible by the figure denoting the root to be extracted, annexing to the decimal part the same number of units so added to the index: then proceed as before. Thus, to divide $\bar{4}$·681241 by 3, write it down thus, $\bar{6}+2$·681241; then $\bar{6}+2$·681241 divided by 3 is $\bar{2}$·893747, the natural number of which is ·078297.

EXAMPLES.

[1]. Required the square root of ·1452, or $\sqrt{\cdot 1452}$.
[2]. Required the 10th root of ·00345, or $\sqrt[10]{\cdot 00345}$.

Calculation.

[1].	[2].
log. ·1452 . . $\bar{1}$·161967	log. . . ·00345 . . $\bar{3}$·537819
or $\bar{2}+1$·161967	or $\overline{10}+7$·537819
log. $\sqrt{\cdot 1452} = \bar{1}$·580983	log. $\sqrt[10]{\cdot 00345} = \bar{1}$·753782
∴ $\sqrt{\cdot 1452} =$ ·381	∴ $\sqrt[10]{\cdot 00345} =$ ·5672

37.	Required the cube root of	·0125	*Ans.*	·2321
38.	. . . square root of	·0093	„	·09644
39.	. . . 72d root of	·096	„	·96797
40.	. . . cube root of	·000048	„	·036342

Rule IX.

To find the value of a number raised to a power represented by a fraction.

(20). Multiply the logarithm of the number by the numerator of the fraction, and divide the result by the denominator; the natural number corresponding to the quotient will be the value required.

EXAMPLES.

[1]. Required the value of $(6\cdot 025)^{\frac{4}{7}}$.
[2]. . . . . $(\cdot 0925)^{1\cdot 5}$, or $(\cdot 0925)^{\frac{3}{2}}$.

Calculation.

[1].

$$\log.\ 6{\cdot}025\ \ldots\ 0{\cdot}779957$$
$$\underline{4}$$
$$7\overline{)3{\cdot}119828}$$
$$\log.\ (6{\cdot}025)^{\frac{4}{7}} = 0{\cdot}445689$$
$$\therefore\ (6{\cdot}025)^{\frac{4}{7}} = 2{\cdot}79054$$

[2].

$$\log.\ {\cdot}0925\ \ldots\ \bar{2}{\cdot}966142$$
$$\underline{3}$$
$$2\overline{)\bar{4}{\cdot}898426}$$
$$\log.\ ({\cdot}0925)^{\frac{3}{2}} = \bar{2}{\cdot}449213$$
$$\therefore\ ({\cdot}0925)^{\frac{3}{2}} = {\cdot}02813$$

Required the value of the following expressions:

41. $({\cdot}096)^{\frac{5}{9}}$. *Ans.* 0·272016

42. $(19)^{\frac{4}{5}}$. „ 10·5439

43. $\dfrac{(466871)^{\frac{6}{7}} \times \sqrt[9]{(3576)^{16}}}{996003 \times \sqrt{{\cdot}0077}}$ „ 1717739

44. The cube root of $({\cdot}001234)^2$ or $({\cdot}001234)^{\frac{2}{3}}$ „ ·011505

45. $(472)^{\frac{5}{9}}$ „ 30·586

46. $({\cdot}042)^{8{\cdot}3}$ ·000000000003741

47. $({\cdot}00563)^{{\cdot}07}$ ·6958825

RULE X.

Of four proportional quantities, any three being given to find the fourth.

(21). If the required quantity be an extreme term, add together the logarithms of the two middle terms, and subtract the logarithm of the other extreme.

If the required quantity be one of the middle terms, add the logarithms of the extremes, and subtract the logarithm of the other middle term.

Thus, if $a : b :: c : x$ and x be required, add together log. b and log. c, and from the sum subtract log. a; the remainder will be log. x.

If $a : b :: x : c$, add together log. a and log. c, and from the sum subtract log. b; the remainder is log. x.

The student will find it convenient in working a proportion by logarithms to make a dash or stroke with the pen under the term *required.*

EXAMPLES.

Given. $6 : 12 : : 50 : x$, to find the value of x.

Calculation.

$$\begin{array}{lr} \text{log. } 12 \ldots\ldots & 1{\cdot}079181 \\ \text{log. } 50 \ldots\ldots & \underline{1{\cdot}698970} \\ & 2{\cdot}778151 \\ \text{log. } \ \ 6 \ldots\ldots & \underline{0{\cdot}778151} \\ \text{log. } \ \ x \ldots\ldots & 2{\cdot}000000 \end{array}$$

$$\therefore x = 100$$

Find the value of x in the following proportions :

48. $24 : 35 : : 79 : x$ *Ans.* $x = 115{\cdot}208$

49. $3505 : x : : 1507 : 29{\cdot}8$ „ $x = 69{\cdot}3$

50. Find a fourth proportional to ·0963, ·24958, and ·008967 *Ans.* ·02324

51. $\sqrt{724} : \sqrt{\frac{5}{13}} : : 6{\cdot}927 : x$ „ ·1596

Rule XI.

*To reduce arithmetical and algebraical expressions to logarithms.**

(22). (*a*) The logarithm of the *product* of any number of terms is equal to the sum of the logarithms of all the terms in the product :

Thus, if $x = ab$, then log. $x =$ log. $a +$ log. b

(*b*) The logarithm of the *quotient* of any two numbers is equal to the logarithm of the dividend diminished by the logarithm of the divisor :

Thus, if $x = a \div b$, or $\frac{a}{b}$, then log. $x =$ log. $a -$ log. b

---

* For the proof of this general rule see Part II.

If $x=\frac{abc}{de}$, then

$$\text{log. } x=\text{log. } a+\text{log. } b+\text{log. } c-\text{log. } d-\text{log. } e$$

(*c*) The logarithm of the *power* of any quantity is equal to the logarithm of the quantity multiplied by the number denoting the power: thus, if $x=a^{10}$, then log. $x=10$ log. a. If $x=a^2b^3$, then log. $x=2$ log. $a+3$ log. b.

(*d*) The logarithm of the *root* of any quantity is equal to the logarithm of the quantity divided by the number denoting the root to be extracted:

Thus, if $x=\sqrt[3]{a}=a^{\frac{1}{3}}$, then log. $x=\frac{\text{log. } a}{3}$ or $\frac{1}{3}$ log. a

If $x=a^{\frac{2}{3}}b^{\frac{1}{3}}$, then log. $x=\frac{2}{3}$ log. $a+\frac{1}{3}$ log. b

Examples of reducing algebraical expressions to logarithms.

Reduce the following expressions to logarithms:

52. $x=abcd$ *Ans.* log. $x=$log. $a+$log. $b+$log. $c+$log. d
53. $x=\frac{ab}{c}$,, log. $x=$log. $a+$log. $b-$ log. c
54. $x=\frac{ab}{cd}$,, log. $x=$log. $a+$log. $b-$log. $c-$log. d
55. $x=a^2bcd^2$,, log. $x=2$ log. $a+$log. $b+$log. $c+2$ log. d
56. $x=\frac{a^2b\sqrt[3]{c}}{10}$,, log. $x=2$ log. $a+$log. $b+\frac{1}{3}$ log. $c-1$
57. $x=\frac{\sqrt{a}.b\sqrt[4]{c}}{d}$,, log. $x=\frac{1}{2}$ log. $a+$log. $b+\frac{1}{4}$ log. $c-$log. d
58. $x=\frac{\sqrt{ab}}{cd^2}$,, log. $x=\frac{1}{2}$ log. $a+\frac{1}{2}$ log. $b-$log. $c-2$ log. d
59. $x=\frac{\sqrt[3]{abc^4}}{\sqrt[4]{d}}$,, log. $x=\frac{1}{3}$log. $a+\frac{1}{3}$log. $b+\frac{4}{3}$log. $c-\frac{1}{4}$log. d

(23). Rules 7, 8, and 9 are derived from the same property of logarithms, namely, log. $a^n=n$ log. a, where n may be any number, either whole or fractional: thus log. $6^5=5$ log. 6; log. $\sqrt[3]{7}=\frac{1}{3}$ log. 7; log. $\cdot 2^{\frac{3}{4}}=\frac{3}{4}$ log. $\cdot 2$.

(24). If the index n of the quantity be *negative*, it will be perhaps the easiest way to reduce it, in the first place, to an equivalent expression with a positive index, which may be always done by substituting for the given quantity its reciprocal,* with the sign of the power changed. For it is proved by Algebra that

$$a^{-n}=\frac{1}{a^n} \text{ and } \frac{1}{a^{-n}}=a^n$$

hence also $x^{-3}=\frac{1}{x^3}$, $4^{-\frac{1}{3}}=\frac{1}{4^{\frac{1}{3}}}$, $\frac{1}{2^{-5}}=2^5$, $\frac{a^{-n}}{b^{-p}}=\frac{b^p}{a^n}$:

we may therefore easily find the logarithms of quantities having negative indices by proceeding as follows: first, reduce the given quantities to equivalent ones with positive indices, and then turn these latter quantities into logarithms by the common rules already given: thus,

$a^{-n}=\frac{1}{a^n}$. Therefore log. $a^{-n}=$ log. $\frac{1}{a^n}=$ log. $1-n$ log. $a=$ $0-n$ log. a (since log. $1=0$).

Again, to reduce $\frac{a^n b^{-p}}{c^{-q}}$ to logarithms:

log. $\frac{a^n b^{-p}}{c^{-q}}=$ log. $\frac{a^n c^q}{b^p}=n$ log. $a+q$ log. $c-p$ log. b:

similarly log. $x^{-3}=$ log. $\frac{1}{x^3}=-3$ log. x; log. $4^{-\frac{1}{3}}=$ log. $\frac{1}{4^{\frac{1}{3}}}=$ $0-\frac{1}{3}$ log. 4; log. $\frac{1}{x^{-5}}=$ log. $x^5=5$ log. x; log. $\frac{a^{-n}}{b^{-p}}=$ log. $\frac{b^p}{a^n}=$ p log. $b-n$ log. a; and so on.

EXAMPLES.

Required the value of the following expressions:

[1] $\cdot 4^{-5}$ [2] $\frac{1}{4^{-5}}$ [3] $\frac{1}{\cdot 2^{-\frac{1}{4}}}$

---

* The reciprocal of any quantity as $\frac{a}{b}$ is $\frac{b}{a}$, the reciprocal of a whole number n, or $\frac{n}{1}$ is $\frac{1}{n}$, and so on.

Calculation.

$\cdot 4^{-5}=\frac{1}{\cdot 4^{5}}$	$\frac{1}{4^{-5}}=4^{5}$	$\frac{1}{\cdot 2^{-\frac{4}{7}}}=\cdot 2^{\frac{4}{7}}$
log. 1 . . 0·000000	log. 4 . . 0·602060	log. ·2 . . $\bar{1}$·301030
log. $\cdot 4^{5}$ . . $\bar{2}$·010300	5	4
log. $\cdot 4^{-5}$ . . 1·989700	log. $\frac{1}{4^{-5}}$ . . 3·010300	$\bar{3}$·204120
$\therefore \cdot 4^{-5}=97\cdot 656$	$\therefore \frac{1}{4^{-5}}=1024$	or $\bar{7}+4\cdot 204120$
		$\therefore$ log. $\cdot 2^{\frac{4}{7}}$ $\bar{1}$·600589
		$\therefore$ $\cdot 2^{\frac{4}{7}}=$ ·3987

Examples to the preceding rules.

Required the value of the following expressions :

60.	3^{7}	*Ans.*	2187
61.	$3^{\frac{1}{7}}$	„	1·16993
62.	3^{-7}	„	·0004572
63.	$3^{-\frac{1}{7}}$	„	·8548
64.	$3^{-\frac{1}{2}}$	„	·5774
65.	$(4\cdot 2)^{\frac{2}{7}}$	„	1·507
66.	$(\cdot 045)^{-\frac{1}{5}}$	„	1·859
67.	$(\cdot 045)^{-\frac{2}{7}}$	„	2·425
68.	$\frac{1}{\cdot 2^{-4}}$	„	·0016
69.	$\frac{1}{(\cdot 02)^{-\frac{4}{5}}}$	„	·04373
70.	$\frac{3^{7}}{9^{2}\times 4^{-6}}$	„	110591·3
71.	$\frac{3^{7}\times 4^{-9}}{3^{-2}}$	„	·07509
72.	$\frac{4^{-5}\times 5^{-\frac{2}{3}}}{35\times\sqrt{2}}$	„	·00000675
73.	$\frac{7\sqrt{15}}{\cdot 015}\times\cdot 0139\sqrt{\frac{2}{\cdot 11}}$	„	107·124

74.	$\dfrac{33\times45\times14}{35\times\sqrt{2}}$	*Ans.* 420·022
75.	$3^{2\frac{1}{2}}$ and $(4^2)^3$	„ 15·5884 and 4096
76.	$(4)^{2^3}$ and $(4)^{2^{\frac{1}{2}}}$	„ 65536 and 7·103
77.	$(50)^{\cdot5^{10}}$	„ 1·00383
78.	$(6)^{\sqrt{5}}$	„ 54·96
79.	$(3124)^{\text{log. }2}$	„ 11·3

(25). Sometimes examples of the following kind occur, in which several of the terms are connected by the signs + or −. In such cases the value of each term must be found separately, and applied according to its sign, as in the following example:

EXAMPLE.

Required the value of $\dfrac{\{6\sqrt{47\cdot5}+3\sqrt[3]{147\frac{1}{2}}\}^{\frac{3}{5}}.(48\frac{3}{4})^{\frac{4}{5}}}{3(1\cdot05)^{20}-100\times\cdot004}$

1. *Calculation of numerator.*

$$\begin{array}{lrl@{\qquad}lrl}
\text{log.} & 47\cdot5\ .. & \underline{1\cdot676694} & \text{log.} & 147\cdot5\ .. & \underline{2\cdot168792} \\
\therefore\text{log.} & \sqrt{47\cdot5}\ .. & 0\cdot838347 & \therefore\text{log.} & \sqrt[3]{147\cdot5}\ .. & 0\cdot722931 \\
\text{log.} & 6\ \ .. & \underline{0\cdot778151} & \text{log.} & 3\ \ .. & \underline{0\cdot477121} \\
\therefore\text{log.} & 6\sqrt{47\cdot5}\ .. & 1\cdot616498 & \therefore\text{log.} & 3\sqrt[3]{147\frac{1}{2}}\ .. & 1\cdot200052
\end{array}$$

$$\therefore\ 6\sqrt{47\cdot5}=41\cdot35 \qquad \therefore\ 3\sqrt[3]{147\tfrac{1}{2}}=15\cdot85$$

$$\text{and}\ \therefore\ 6\sqrt{47\cdot5}+3\sqrt[3]{147\cdot5}=57\cdot2$$

$$\begin{array}{lrl@{\qquad}lrl}
\text{log.} & 57\cdot2\ .. & 1\cdot757396 & \text{log.} & 48\cdot75\ .. & 1\cdot687975 \\
 & & \underline{3} & & & \underline{4} \\
 & & 5)\underline{5\cdot272188} & & & 5)\underline{6\cdot751900} \\
\text{log.} & (57\cdot2)^{\frac{3}{5}}\ .. & 1\cdot054437 & \text{log.} & (48\frac{3}{4})^{\frac{4}{5}}\ .. & 1\cdot350380 \\
 & & & \text{and log.} & (57\cdot2)^{\frac{3}{5}}\ .. & \underline{1\cdot054437} \\
 & & & & \therefore\ \text{log. numerator} & =2\cdot404817
\end{array}$$

2. *Calculation of denominator.*

$$\begin{array}{ll}
\text{log. } 1\cdot05\ \ldots\ldots\ldots & 0\cdot021189 \\
 & \underline{20} \\
\text{log. } (1\cdot05)^{20}\ldots\ldots & 0\cdot423780 \\
\text{log. } 3\ \ldots\ldots\ldots\ldots & \underline{0\cdot47712} \\
\textit{log. } 3\ (1\cdot05)^{20}\ldots & 0\cdot900901
\end{array}$$

$\therefore\ 3\,(1{\cdot}05)^{20}=7{\cdot}96$	log. numerator ...... $2{\cdot}404817$
and $100\times{\cdot}004=\ {\cdot}40$	log. denominator ... $0{\cdot}878522$
$\therefore$ denominator$=7{\cdot}56$	$\therefore$ log. fraction ...... $1{\cdot}526295$
	and fraction$=33{\cdot}6$ *Ans.*

Required the value of the following expressions:

80. $$\frac{2(\frac{3}{2})^{15}-2}{\frac{3}{2}-1}$$ *Ans.* 1747·6

81. $$\left(\frac{\frac{5}{9}\cdot\sqrt[18]{2}}{\frac{10}{17}\cdot\sqrt[23]{3}}+5\right)^3$$ " 94·794

82. $$\frac{\sqrt[5]{\frac{7}{3}\sqrt[4]{6}}+253\sqrt[3]{\frac{716{\cdot}5}{\sqrt{2}}}-20}{19^{\frac{4}{5}}-4{\cdot}54}$$ " 333·03

(26.) By means of the preceding rules the following equations may be solved.

Find an approximate value of x in the following equations:

83. $10^x=456$ *Ans.* $x=2{\cdot}658965$

84. $x^3=14$ " $x=2{\cdot}4101$

85. $x^5=14{\cdot}76$ " $x=1{\cdot}71323$

86. $3^{2x}=20$ " $x=1{\cdot}3634$

87. $x^3={\cdot}004$ " $x={\cdot}1587$

88. $x^{-3}=4\frac{1}{2}$ " $x={\cdot}6057$

89. $x=({\cdot}02445)^{\frac{3}{4}}$ " $x={\cdot}06183$

90. $x=\sqrt[2000]{47691}$ " $x=1{\cdot}0054$

91. $x=\sqrt{\dfrac{2}{123}}$ " $x={\cdot}1275$

92. $x=\sqrt[3]{\dfrac{1}{3{\cdot}14159}}$ " $x={\cdot}6827$

93. $a^x=b$ " $x=\dfrac{\log. b}{\log. a}$

94. $\dfrac{a^x}{b^x}=c$ " $x=\dfrac{\log. c}{\log. a-\log. b}$

95. $\dfrac{a^{mx}}{b^{nx-1}}=c$ " $x=\dfrac{\log. c-\log. b}{m\log. a-n\log. b}$

96. $a^x=\frac{b^{mx-n}}{c^{rx}}$ *Ans.* $x=\frac{n \log. b}{m \log. b-\log. a-r \log. c}$

97. $(\frac{5}{4})^x=54\frac{1}{2}$ „ $x=17\cdot91$

98. $a^{\frac{b}{x}}=c$ „ $x=\frac{b \log. a}{\log. c}$

99. $a^{b^x}=c$ „ $x=\frac{1}{\log. b}.\log.\left(\frac{\log. c}{\log. a}\right)$

100. $(10\frac{1}{2})^x=20$ „ $x=1\cdot27$

101. $2^{3^x}=4$ „ $x=\cdot6309$

Rule XII.

To find the value of a fractional logarithmic equation, when the index of the logarithm of one or both of the terms is negative.

Reduce the logarithm with the negative index to a quantity wholly negative (Rule II. p. 5). The logarithm then becomes wholly negative, and the division can be performed by the common algebraical rule.

EXAMPLE.

Given $(\cdot04)^{3x}=5$; find the numerical value of x.

Since $(\cdot04)^{3x}=5$ $\therefore 3x \log. \cdot04=\log. 5$

$$3x=\frac{\log. 5}{\log. \cdot04}=\frac{0\cdot698970}{\bar{2}\cdot602060}=\frac{\cdot69897}{-1\cdot39794}=-0\cdot5$$

$$\therefore x=-0\cdot1666$$

Find the numerical value of x in the following equations:

102. $4^{3x}=\cdot005$ *Ans.* $x=-1\cdot274$

103. $(\cdot04)^{3x}=\cdot001$ „ $x=\cdot715$

104. $(\cdot04)^{x^2}=5$ „ $x=\sqrt{-(0\cdot5)}$

105. $(\cdot04)^{\sqrt{x}}=5$ „ $\sqrt{x}=-0\cdot5$

LOGARITHMIC SINES, &c. NATURAL VERSINES.

(27). In addition to the table of logarithms of numbers, the use of which has been illustrated by so many examples, the student must make himself acquainted with the table of log. sines, tangents, &c.; as by means of these two tables nearly all the problems in Trigonometry and Navigation are solved.

Another table, called the table of natural versines, is sometimes used with advantage in questions where an angle or an arc is required to the nearest second; since, this table being computed to seconds, the value of the arc or angle may be taken out by inspection, and thus the tedious labour of proportioning for seconds, which it is sometimes necessary to do when the table of log. sines is used, is avoided.

As rules and directions for taking quantities out of these tables to the nearest second accompany the tables, we will give in this place only a few examples as exercises.

To take out quantities from table of log. sines, tangents, &c.

EXAMPLES.

106. Required the log. sine, log. tangent, and log. cosecant of the following angles (to the nearest second):

Angles.		Log. sine.	Log. tan.	Log. cosec.
10° 10′ 6″	*Ans.*	9·246845	9·253720	10·753155
19 10 40	„	9·516536	9·541332	10·483464
48 35 35	„	9·875079	10·054613	10·124921
61 24 40	„	9·943532	10·263630	10·056468

107. Required the angles (to the nearest second) whose log. sines are:

9·641452	*Ans.*	25° 58′ 30″
9·714185	„	31 11 12
9·984204	„	74 38 26

In the above examples the value of the angle is found to the *nearest second; but in* the common problems in Navigation it is seldom necessary to proportion to the nearest second, it being sufficiently correct to take out the quantity required *by inspection.* In Inman's Nautical Tables, the log. sine, &c. of an angle can be taken out by inspection to the nearest 15″; in nearly all other Nautical Tables to the nearest minute.

(28). When the angle whose trigonometrical ratio is required is greater than 90°, its value may be found in the table as follows: Subtract the angle from 180°, and look for the remainder, which is called its *supplement,* in the tables. Thus, to find the log. sine of 100° 10′, subtract it from 180°, and look for the log. sine of the remainder (namely, 79° 50′), which is 9·993127; therefore log. sine 100° 10′= 9·993127.

(29). But the readiest way will, in general, be to *diminish* the given angle by 90°, and to look out the remainder according to the following rule:

Rule XIII.

To take out from the table the log. sine, log. cosine, &c. of an angle greater than 90°.

Diminish the angle by 90°: then
if sine is required look out *co*sine of remainder
*co*sine . . . . . sine . . . . .
tangent . . . . . *co*tangent . . .
and in the same manner for secant, cosecant, &c.

EXAMPLES.

108. Take out of the tables the log. cosine of 100° and the log. cosecant of 170° 14′ 15″.

log. cos. 100°=log. sin. 10°
log. cosecant 170° 14′ 15″=log. sec. 80° 14′ 15″

By the tables:

log. sin. 10°=9·239670 and
log. secant 80° 14′ 15″=10·770665
∴ log. cos. 100°=9·239670 and
log. cosecant 170° 14′ 15″=10·770665.

(30). *Table of natural versines.*

EXAMPLES.

109. Find the natural versines of the following angles:

26° 32′ 15″	*Ans.*	105357
157 48 50	„	1925964
90 7 15	„	1002109
125 0 30	„	1573695

110. Required the angles whose natural versines are:

1175443	*Ans.*	100° 6′ 16″
105357	„	26 32 15
1925964	„	157 48 50
1573695	„	125 0 30

There are other special tables used in Trigonometry and Navigation, but they require no examples in this place to illustrate their use.

(31). *Reduction of trigonometrical formulæ to tabular logarithms.*

Trigonometrical formulæ are reduced to logarithms by the common rules for reducing algebraical formulæ (22); but in order to avoid, as much as possible, the inconvenience of using negative indices in calculations, the table of log. sines, cosines, &c. is constructed by adding 10 to the indices of the log. sines, &c. Thus, the sine 30° is proved in Trig. Part II. to be equal to ½ or ·5, its logarithm must therefore be $\bar{1}$·698970: but the log. sine of 30° contained in the tables, or as it is called the *tabular* log. sine, is 9·698970; that is, the tabular log. sine is equal to the log. sine + 10, and therefore the log. sine of any angle = tab. log. sine − 10. And as the same addition is made to the indices of the logs. of all the trigonometrical ratios, we must, in reducing such trigonometrical formulæ to tabular logarithms, *subtract* 10 *from the logarithm of each trigonometrical ratio used in the expression.*

For a similar reason, viz. to avoid the use of negative indices, the table of natural versines is constructed by multiplying the natural versines by one million; thus, it is shown in Part II. that versine $60° = \frac{1}{2} = \cdot 5$; but the versine of 60° as contained in the table $= \cdot 5 \times 1000000 = 500000$; that is, tab. vers. = ver. × 1000000; therefore, in logarithms,

$$\text{log. tab. vers.} = \text{log. vers.} + 6$$
$$\text{and log. vers.} = \text{log. tab. vers.} - 6$$

Hence we have the following rule for reducing trigonometrical formulæ to tabular logarithms.

RULE XIV.

To reduce a trigonometrical formula to tabular logarithms.

Subtract 6 from each log. versine, and 10 from the logarithm of each of the other trigonometrical terms that occur in the formula.

EXAMPLES.

111. Thus, if tan. A = sin. A . sec. A

In tabular logarithms,

$$\text{tab. log. tan. A} - 10 = \text{tab. log. sin. A} - 10 + \text{tab. log. sec. A} - 10$$

or, as it is usually written (suppressing the word tabular, it being understood that the formula is reduced to *tabular* logarithms),

$$\text{log. tan. A} - 10 = \text{log. sin. A} - 10 + \text{log. sec. A} - 10$$

and collecting and cancelling the tens,

$$\text{log. tan. A} = \text{log. sin. A} + \text{log. sec. A} - 10$$

If $\text{tan. A} = \frac{\text{sin. A}}{\text{cos. A}}$ In tab. logs.

$$\text{log. tan. A} - 10 = \text{log. sin. A} - 10 - (\text{log. cos. A} - 10)$$
$$\text{or log. tan. A} = \text{log. sin. A} + 10 - \text{log. cos. A}$$

112. If $\text{ver. } x = 2 \text{ sin. } b \text{ . sin. } c \text{ . sin. }^2 \frac{A}{2}$ In tab. logs.

$$\text{log. ver. } x - 6 = \cdot 301030 + \text{log. sin. } b - 10 + \text{log. sin. } c - 10 + 2 \text{ log. sin. } \frac{A}{2} - 20$$

$$\text{or log. vers. } x = 6 \cdot 301030 + \text{log. sin. } b + \text{log. sin. } c + 2 \text{ log. sin. } \frac{A}{2} - 40$$

EXAMPLES.

Reduce to tabular logarithms the following formulæ:

113. $\sin. x = \text{cosec}. y . \tan. z$ 114. $a = b . \tan. A$

115. $\tan. A = \frac{a}{b}$ 116. $\frac{x}{a} = \cot. A$

Answers to 113-6.

113. $\log. \sin. x = \log. \text{cosec}. y + \log. \tan. z - 10$
114. $\log. a = \log. b + \log. \tan. A - 10$
115. $\log. \tan. A = 10 + \log. a - \log. b$
116. $\log. x = \log. a + \log. \cot. A - 10$

In almost every problem in Trigonometry and Navigation we shall have occasion to reduce trigonometrical formulæ to tabular logarithms. We will therefore give in this place a few more examples for reduction, and also for finding the value of the unknown quantity by assigning numerical values to the given quantities. These examples will be found of great use.

117. Given $\frac{x}{a} = \cot. A$; find the numerical value of x when $a = 20$, and angle $A = 30° 10'$.

Since $\frac{x}{a} = \cot. A \therefore x = a \cot. A$

In tab. log. . . . . $\log. x = \log. a + \log. \cot. A - 10$.

Calculation.

$a = 20$	log. a.	1·301030
$A = 30° 10'$	log. cot. A. .	10·235648
		11·536678
		10
	log. x.	1·536678

$\therefore x = 34·41$

118. Given $\tan. x = \frac{\sin. A \cos. B}{\tan. C}$; to find x, when $A = 20° 10'$, $B = 32° 45'$, and $C = 78° 45'$.

$$\text{tan. } x=\frac{\text{sin. A cos. B}}{\text{tan. C}}\quad \text{In tabular logarithms,}$$

$$\begin{aligned}\text{log. tan. } x-10&=\text{log. sin. A}-10+\text{log. cos. B}-10-(\text{log. tan. C}-10)\\&=\text{log. sin. A}-10+\text{log. cos. B}-10-\text{log. tan. C}+10\end{aligned}$$

Cancelling the tens, we have

$$\text{log. tan. } x=\text{log. sin. A}+\text{log. cos. B}-\text{log. tan. C}$$

A=20° 10′	log. sin. A . .	9·537507
B=32 45	,, cos. B . .	9·924816
C=78 45		19·462323
	,, tan. C . .	10·701338
	,, tan. x . .	8·760985
		∴ $x=3°\ 18'$

119. In the equation, $x=a$. tan. A . sin. B . cosec. C, find x, having given $a=416$ feet, A=23° 50′ 15″, B=54° 28′ 30″, and C=147° 32′ 50″. *Ans.* $x=278{\cdot}7$ feet.

RULE XV.

To find the natural versine of an angle by means of the common table of logarithmic sines, &c.

First. When the angle is less than 90°.

Take out the log. cosine of the given angle and diminish the index by 10, and find the natural number corresponding to the remainder as a logarithm: this will be the natural cosine of the angle. Subtract this natural cosine from unity: the result will be the natural versine of the given angle, which, multiplied by one million, will reduce it to the usual tabular versine.

Second. When the angle is greater than 90°.

Subtract 90° from the given angle. Take out the log. sine of the result and diminish its index by 10: then find, as before, the natural number of the remainder. Add unity to the natural number, and the sum will be the natural versine of the given angle. To get the usual tabular versine, multiply *the natural* versine by 1000000.

120. EXAMPLE 1. Find the tabular natural versine of 26° 32′ 15″.

tabular log. cos. 26° 32′ 15″ ...... 9·951650
10
log. cos. 26° 32′ 15″ .... $\bar{1}$·951650
∴ natural cos. 26° 32′ 15″ ...... 0·894643
1·
∴ natural versine = 0·105357

and tab. nat. versine = 105357.

121. EXAMPLE 2. Find the tabular natural versine of 125° 0′ 30″.

tabular log. sine 35° 0′ 30″......... 9·758681
10
log. cos. 125° 0′ 30″ .... $\bar{1}$·758681
∴ natural cos. 125° 0′ 30″ ...... 0·573695
1
∴ natural versine = 1·573695

and tab. nat. versine = 1573695.

(32). *Use of the algebraic signs + and − to determine the magnitude of an angle.*

The numerical value of any trigonometrical ratio of an angle and of its supplement (28) being the same, therefore the log. of any trigonometrical ratio taken out of the tables corresponds to both angles: thus, sin. 100° being equal to sin. 80°, the log. sine of 100° = log. sine of 80°. For this reason, when we have given the log. sine of an angle to find the angle, we are not always certain that the angle found in the tables or its supplement is the correct one. This uncertainty, however, can be removed when the trigonometrical ratio in question is a tangent, cotangent, secant, or cosine, by finding, by the following Rule, whether its algebraic sign is + or −. When the algebraic sign is +, the angle taken out *of the tables is the* one sought: when the algebraic sign is —, we must subtract the angle taken out from 180° for the angle required.

RULE XVI.

(33). *To determine the algebraic sign of a trigonometrical ratio in any formula adapted to logarithms where all the other terms are known, and thence to find the magnitude of the angle, that is, whether it is greater or less than* 90°.

Write down the formula, and simplify it by clearing it of fractions, &c.; put over each given term its proper algebraical sign; that is, over each tangent, secant, cosine, and cotangent of an angle or arc less than 90°, the sign + ; and over each of the same quantities when the angle is greater than 90°, the sign —, but over each sign, cosecant, and versine, the sign +, whether the angle is greater or less than 90°; and determine from thence the sign of the product of that side whose terms are all known.

Then, since the sign of the product of each side of the equation must be the same (otherwise we should have a positive quantity equal to a negative quantity), make it so by putting over the unknown term the sign + or — accordingly.

Then, if + falls over the unknown term, the part required is less than 90°, and the quantity taken out will be the angle required; but if —, subtract the angle taken out of the table from 180°, the remainder will then be the angle required.

(*Ambiguous case*). When the part sought is expressed in terms of the sine, the above rule will not apply, since the sine is positive, whether the angle is greater or less than 90°. The uncertainty which thence arises (forming what is called in Trigonometry the *ambiguous case*) can be removed only in particular cases.

In the above rule the angle required is supposed to be less than 180°.

EXAMPLES.

In each of the following formulæ it is required to find the value of x: supposing $A=45°$, $B=120°$, and $C=130°$.

122. Given $\underline{\text{cos. } x}=\text{tan. A . cos. B . sec. C.}$
Find in the first place whether x is greater or less than 90°.

By the rule we see that tan. A is +, cos. B is −, and sec. C is − ; placing, therefore, over the given terms their proper signs, we have

$\text{cos. } x=\overset{+}{\text{tan.}}\text{ A . }\overset{-}{\text{cos.}}\text{ B . }\overset{-}{\text{sec.}}\text{ C}$; and since the product of the three terms on the right-hand side of the equation is positive, cos. x must also be positive ; put the sign + over cos. x accordingly, which shows that x must be less than 90° : hence the formula becomes $\underline{\overset{+}{\text{cos.}}\ x}=\overset{+}{\text{tan.}}\text{ A . }\overset{-}{\text{cos.}}\text{ B . }\overset{-}{\text{sec.}}\text{ C.}$

Calculation.

$$\text{log. cos. } x-10=\text{log. tan. A}-10+\text{log. cos. B}-10+\text{log. sec. C}-10$$
$$\text{or log. cos. } x=\text{log. tan. A}+\text{log. cos. B}+\text{log. sec. C}-20$$

A=45°	log. tan. A ...	10·000000
B=120	„ cos. B ...	9·698970
C=130	„ sec. C ...	10·191933
	„ cos. x ...	9·890903

∴ $x=38°\ 56'$

123. Given $\underline{\text{sec. } x}=\text{sin. A . sec. B.}$

Find whether x is greater or less than 90°.

Determining by the rule the signs of sin. A and sec. B, we see that the sign − must be placed over sec. x, or

$\overset{-}{\text{sec.}}\ x=\overset{+}{\text{sin.}}\text{ A . }\overset{-}{\text{sec.}}\text{ B}$, a negative product; therefore sec. x is negative, or x is greater than 90°.

Calculation.

$$\text{log. sec. } x=\text{log. sin. A}+\text{log. sec. B}-10.$$

A=45°	log. sin. A ...	9·849485
B=120	„ sec. B ...	0·301030 (rejecting 10)
	„ sec. x ...	10·150515
		45°
		180

∴ $x=135$

124. Given $\tan. x = \text{cosec. A} \,.\, \cos. B$.

Find whether x is greater or less than 90°.

Placing over cosec. A and cos. B their proper algebraic signs, we see that tan. x is negative, or

$$\overset{-}{\underline{\tan. x}} = \overset{+}{\text{cosec. A}} \,.\, \overset{-}{\cos. B}$$

and therefore x is greater than 90°.

125. $$\text{cosec. A} \,.\, \sin. B = \underline{\cot. x}$$

Proceeding in a similar manner, we find that cot. x is positive, and therefore x is less than 90°.

126. $$\sec. A \,.\, \sin.^2 C = \cos.^2 B \,.\, \underline{\cot. x}.$$

Placing over the given quantities their proper signs, and recollecting that the square of $\overset{-}{\cos. B}$. must be positive, we have

$$\overset{+}{\sec. A} \,.\, \overset{+}{\sin.^2 C} = \overset{+}{\cos.^2 B} \,.\, \overset{+}{\underline{\cot. x}}$$

$$\therefore x \text{ is less than } 90°.$$

127. $$\sec. A \,.\, \sin.^2 C = -\cos.^2 B \,.\, \underline{\cot. x}$$

Placing the signs over the given quantities, and taking into consideration the negative sign in front of the right-hand side of the equation, we see that — must be placed over cot. x in order to render the right-hand side of the equation the same sign as the left, namely, positive : hence

$$\overset{+}{\sec. A} \,.\, \overset{+}{\sin.^2 C} = -\overset{+}{\cos.^2 B} \,.\, \overset{-}{\underline{\cot. x}}$$

$$\therefore x \text{ is greater than } 90°.$$

128. $$\overset{+}{\sin. A} \,.\, \overset{-}{\underline{\cos. x}} = -\overset{-}{\cos. B} \,.\, \overset{-}{\cot. C}$$

$$\therefore x \text{ is greater than } 90°.$$

129. $$\text{ver. A} = \frac{\cos. B}{\sin. C \,.\, \tan. x}$$

$$\text{or } \overset{+}{\text{ver. A}} \,.\, \overset{+}{\sin. C} \,.\, \overset{-}{\underline{\tan. x}} = \overset{-}{\cos. B}$$

or x is greater than 90°.

(34). Sometimes trigonometrical expressions occur in which several of the terms are connected by the signs + or −, and which cannot be reduced to a convenient logarithmic form. To determine the numerical value of such expressions, the value of each term must be found separately, and then applied with its proper sign. The method of proceeding will be seen in the following example:

130. Given $u=\sqrt[3]{\tan x\,.\,\sec.\, y-a^2b\,.\,\cos.^2 y}$; to find the value of u, when $x=32°\ 10'$, $y=80°$, $a=25$, and $b=50$.

Ans. $u=-9·79$.

Calculation.

Let $m=\tan.\ x\ \sec.\ y.$ $\quad n=a^2b\ \cos.^2 y$

$\therefore\ u=\sqrt[3]{m-n}$

log. m=log. tan. x+log. sec. $y-20$

log. n=2 log. a+log. b+2 log. cos. $y-20$

and log. $u=\frac{1}{3}$ log. $(m-n)$

To find m.

log. tan. x .... 9·798596

log. sec. y .... 10·760330

log. m .... 0·558926

$\therefore\ m=3·62$

To find u.

log. $u=\frac{1}{3}$ log. $(m-n)$

log. $(m-n)$ .... 2·972517

$\therefore\ \frac{1}{3}$ log $(m-n)$... 0·990839

$\therefore\ u=-9·79$

To find n.

log. a ...... 1·397940

log. a ...... 1·397940

log. b ...... 1·698970

log. cos. y.. 9·239670

log. cos. y.. 9·239670

log. n ...... 2·974190

$\therefore\ n=$ 942·3

and $m=$ 3·62

$\therefore\ m-n=-938·68$

Find u in the following expressions: the values of a, b, x, and y, being the same as in the last example.

131. $u=a\ \cos.\ x-b\ \sin.\ y.$ *Ans.* $u=-28·08$

132. $u=\dfrac{a\ \sec.\ x+b\ \cos.^2 y}{\tan.\ x-\cos.\ y}.$ „ $u=$ 68·1

Logarithms *were originally* invented to facilitate trigonometrical calculations. The following examples will show their use in problems of another kind.

(35). *Application of logarithms to questions in progression, interest, mensuration, &c.*

IN GEOMETRICAL PROGRESSION. If a=the first term of a geometrical series, r=common ratio, n=number of terms, and S=sum of terms; then it is proved in Algebra that $S=a\,.\,\frac{r^n-1}{r-1}$. Any three of these quantities being given, the fourth may be readily found by means of logarithms.

EXAMPLES.

133. Find the sum of 20 terms of the series.

$$1,\ \tfrac{3}{2},\ \tfrac{9}{4},\ \tfrac{27}{8}\ \ldots\ldots$$

$$S=a\,.\,\frac{r^n-1}{r-1}=1\,.\,\frac{(\frac{3}{2})^{20}-1}{\frac{3}{2}-1}=\frac{(1{\cdot}5)^{20}-1}{\frac{1}{2}}$$

$$\log.\ (1{\cdot}5)^{20}=20\ \log.\ 1{\cdot}5=3{\cdot}521820$$

$$\therefore\ (1{\cdot}5)^{20}=3325{\cdot}21$$

and $S=\frac{3325{\cdot}21-1}{\frac{1}{2}}=\frac{3324{\cdot}21}{\frac{1}{2}}=6648{\cdot}42$. *Ans.*

134. The sum of a geometrical series is 6560, its first term 2, and common ratio 3 : find the number of terms.

$$S=a\frac{r^n-1}{r-1}\qquad\therefore\ 6560=2\,.\,\frac{3^n-1}{3-1}=3^n-1$$

$$\therefore\ 3^n=6561,\text{ and } n\log.3=\log.6561,\ \therefore\ n=\frac{\log.\ 6561}{\log.\ 3}=8$$

135. If sum of series=1023, first term=1, common ratio =2 ; what is the number of terms? *Ans.* $n=10$.

Without the use of logarithms the last two questions would be very difficult to answer.

IN COMPOUND INTEREST. If P denote the principal, r the interest of £1 for 1 year, n the number of years, and A the amount, then it is proved in Algebra that $A=P\,(1+r)^n$: any three of these quantities being given, the fourth may be found.

EXAMPLES.

136. If £200 be placed out at compound interest for 7 years at 4 per cent: required the amount.

$$A=P(1+r)^n$$

$\therefore$ log. A=log. P+n log. $(1+r)$=log. 200+7 log. (1+·04)

Ans. A=£263 3*s.* 8*d.*

137. At what rate of interest must £400 be placed out that it may amount to £569 6*s.* 8*d.* in nine years, at compound interest?

$$A=P(1+r)^n \qquad \therefore \log.\,(1+r)=\frac{\log.\,A-\log.\,P}{n}$$

Ans. r=·04, or 4 per cent.

138. In how many years will £500 amount to £900 at 5 per cent, compound interest?

$$A=P(1+r)^n \qquad \therefore n=\frac{\log.\,A-\log.\,P}{\log.\,(1+r)}$$

Ans. n=12·04 years.

139. In Astronomy. It is proved that the *squares* of the times which two planets employ to make their revolutions about the Sun are to each other as the *cubes* of their distances from the same heavenly body. Knowing that the revolution of the Earth about the Sun is performed in 365 days, 5 hours, 48 minutes, 51 seconds; and that of Jupiter in 4330 days, 14 hours, 39 minutes, 2 seconds: it is required to find the ratio of the distances of these planets from the Sun.

Reducing in the first place into seconds the given revolutions, we have 31556931 seconds for the Earth, and 374164742 for Jupiter. Representing the first by a, the second by b, the distance of the Earth from the Sun by 1, and that of Jupiter by x, we have

$$x^3 : 1^3 :: b^2 : a^2 \qquad \text{or } x^3=\frac{b^2}{a^2}$$

$$\therefore 3 \log.\, x=2 \log.\, b-2 \log.\, a$$

$$\text{and } \log.\, x=\frac{2 \log.\, b-2 \log.\, a}{3}=0{\cdot}715978 \qquad \therefore x=5{\cdot}1997$$

Or the *distance of Jupiter* from the Sun is to that of the *Earth from the same body* nearly as 52 : 10.

Formulæ for the principal regular bodies.

$\dfrac{\text{circumference of circle}}{\text{diameter}}=\pi$ where $\pi=3{\cdot}1416$ nearly.

Area of rectangle $=a\,b$. a, b, adjacent sides.

,, parallelogram $=ab$ sin. C a, b, sides, C included angle

,, triangle $=\sqrt{s\,.\,\overline{s-a}\,.\,\overline{s-b}\,.\,\overline{s-c}}$ a, b, c, sides, and $s=\frac{1}{2}(a+b+c)$

,, triangle $=\frac{1}{2}\,a\,b$ sin. C a, b, sides, C included angle

,, regular polygon $=\frac{1}{4}\,n\,a^2$ cot. $\dfrac{180^\circ}{n}$ n sides each $=a$

,, circle $=\pi r^2=\dfrac{\pi}{4}d^2$ $r=$ radius $d=$ diameter

,, annulus $=\dfrac{\pi}{4}(d^2-d_1{}^2)=\dfrac{\pi}{4}(d+d_1)\,(d-d_1)$
exterior and interior diameters d and d_1

,, ellipse $=\dfrac{\pi}{4}dd_1$ major and minor diameters d and d_1

Surface (convex) of cylinder $=\pi d\,h$ dia. d height h

,, cone $=\dfrac{\pi}{2}d\,h_1$ dia. d slant height h_1

,, parabola $=\frac{4}{3}xy=\frac{2}{3}$ of parallelogram of same height and base, coordinates x, y

,, sphere $=\pi d^2=cd$ dia. d circumf. c

Volume of rectangular parallelopiped $=a\,b\,c$ a, b, c, sides

,, cylinder $=\dfrac{\pi}{4}d^2\,h=\dfrac{c^2h}{4\pi}={\cdot}079577\;c^2\,h$
dia. d height h circumf. c

,, cone $=\frac{1}{3}\cdot\dfrac{\pi}{4}d^2h={\cdot}2618d^2h={\cdot}026528c^2\,h$
dia. d height h circumf. c

,, sphere $=\frac{1}{6}\pi d^3={\cdot}5236d^3$ dia. d.

,, prolate spheroid $=\frac{2}{3}\times$ circumscribing cylinder $=$ $\frac{4}{3}\,\pi\,a\,b^2$ a semi-major, b semi-minor axis

Volume of oblate spheroid $=\frac{2}{3}\times$ circumscribing cylinder $=$ $\frac{4}{3}\,\pi\,a^2\,b$ a semi-major, b semi-minor axis

,, paraboloid $=\frac{1}{2}\times$ circumscribing cylinder
$=\frac{1}{2}\cdot\dfrac{\pi}{4}d^2\,h$ dia. d height h

Mensuration of areas, surfaces, and solids.

The following examples in mensuration are given chiefly as exercises in logarithms: they can all be easily calculated by means of the formulæ contained in page 36.

EXAMPLES.

140. Two adjacent sides of a rectangle are $40\frac{1}{2}$ feet and $28\frac{3}{4}$ feet: required its area.

area $= a\,b$ (p. 36).

$a = 40\frac{1}{2} = 40{\cdot}5$ $\quad\therefore$ log. area $=$ log. $a +$ log b

$b = 28\frac{3}{4} = 28{\cdot}75$

log. a .... $1{\cdot}607455$

log. b .... $1{\cdot}458638$

log. area ..... $3{\cdot}066093$

$\therefore$ area $= 1164{\cdot}375$ sq. feet.

141. Two adjacent sides of a rectangular field are $675{\cdot}5$ yards and $473{\cdot}3$ yards: required its area or superficial content.

Ans. $319712{\cdot}96$ sq. yards.

142. The three sides a, b, c, of a plane triangle ABC are $a = 1150$, $b = 937{\cdot}5$, and $c = 687{\cdot}5$ yards: required its area or superficial content.

$a = 1150$ $\quad$ Area $= \sqrt{s \cdot \overline{s-a} \cdot \overline{s-b} \cdot \overline{s-c}}$ (p. 36).

$b = 937{\cdot}5$ $\quad\therefore$ log. area $= \frac{1}{2}\{$log. $s +$ log. $\overline{s-a} +$ log. $\overline{s-b}$

$c = 687{\cdot}5$ $\quad + $ log. $\overline{s-c}\}$

$a+b+c$ .. $2775{\cdot}0$

$\frac{1}{2}(a+b+c)$ $1387{\cdot}5 = s$ $\qquad$ log. s .... $3{\cdot}142233$

$\therefore$ $237{\cdot}5 = s-a$ $\qquad$,, $s-a$ ...... $2{\cdot}375664$

$450{\cdot}0 = s-b$ $\qquad$,, $s-b$ ...... $2{\cdot}653213$

$700{\cdot}0 = s-c$ $\qquad$,, $s-c$ ...... $2{\cdot}845098$

$2)\underline{11{\cdot}016208}$

,, area ...... $5{\cdot}508104$

$\therefore$ area $= 322185$ sq. yds.

143. The three sides of a triangular field are $a=376{\cdot}2$, $b=450{\cdot}25$, $c=502{\cdot}25$: required its superficial content.

Ans. 81509·3

144. Two sides a and b of a plane triangle ABC are $a=52{\cdot}25$ yards and $b=60{\cdot}125$ yards, and the included angle $C=72^\circ\ 32'$: required its area.

$$\text{Area}=\tfrac{1}{2}\,ab\ \text{sin. C.} \qquad \text{(p. 36).}$$

$$\begin{array}{lll}
a=52{\cdot}25 & \text{log. area}=\text{log. }a+\text{log. }b+\text{log. sin. C}-10-\text{log. }2 & \\
b=60{\cdot}125 & \text{log. }a\ \ldots & 1{\cdot}718086 \\
C=72^\circ\,32' & \text{log. }b\ \ldots & 1{\cdot}779055 \\
 & \text{log. sin. C} & \underline{9{\cdot}979499} \\
 & & 13{\cdot}476640 \\
 & & \underline{10{\cdot}301030} \\
 & \text{log. area} & 3{\cdot}175610
\end{array}$$

$$\therefore\ \text{area}=1498{\cdot}34\ \text{sq. yards.}$$

145. Two sides of a plane triangle ABC are $b=812{\cdot}125$ $c=1015{\cdot}25$, and angle $A=57^\circ\ 35'$: find its superficial content.

Ans. 348014·8

From the two formulæ used above are deduced Rules VI. and VII. in Plane Trigonometry given hereafter for finding the area of a triangle.

146. A side of a regular heptagon is 19·38 yards: required its area.

$$\text{area}=\tfrac{1}{4}n\,a^2\,.\,\text{cot.}\frac{180^\circ}{n} \qquad \text{(p. 36).}$$

$$a=19{\cdot}38 \qquad n=7 \qquad \frac{180^\circ}{7}=25^\circ\ 42'\ 51''$$

$$\therefore\ \text{log. area}=\text{log. }n+2\ \text{log. }a+\text{log. cot.}\frac{180}{n}-10-\text{log. }4$$

log.	n ...	0·845098
„	a ...	1·287354
„	a ...	1·287354
„	cot. $\frac{180^\circ}{n}$	0·317339 (rejecting 10 in index)
		3·737145
„	4 ...	0·602060
„	area ...	3·135085

$\therefore$ area$=$1364·8 sq. yds.

147. A side of an eleven-sided regular polygon is 20 feet; required the area or surface.

Ans. 3746·27 sq. feet.

It is evident that in the above and similar formulæ any one of the terms may be found, if all the rest are given: thus, in the formula (p. 36) area$=\frac{1}{4}\,n\,a^2$ cot. $\frac{180^\circ}{n}$, let there be given the area$=$3746·27 sq. feet and $n=11$; to find a, a side of the regular polygon.

Thus, since area$=\frac{1}{4}\,na^2$ cot. $\frac{180^\circ}{n}$

$\therefore a=\sqrt{\dfrac{4 \text{ area}}{n \text{ cot. } \frac{180^\circ}{n}}}$: reducing this expression to logarithms we have (Rule XIV. p. 26)

$$\text{log. } a=\tfrac{1}{2}\left\{10+\text{log. } 4+\text{log. area}-(\text{log. } n+\text{log. cot. }\frac{180^\circ}{n})\right\}$$

whence $a=20$

Again, in the formula area$=\frac{1}{2}\,ab$ sin. C, let there be given the area$=$1498·34 sq. yds., $a=52\frac{1}{4}$ yds., and $b=60\frac{1}{8}$ yds., to find the angle C included between the two given sides.

Reducing the formula to logarithms we have

$$\text{log. area}=\text{log. } a+\text{log. } b+\text{log. sin. C}-10-\text{log. } 2$$

$$\therefore \text{log. sin. C}=10+\text{log. } 2+\text{log. area}-(\text{log. } a+\text{log. } b)$$

whence $C=72^\circ\ 32'$

148. The diameter of a circle is $1\frac{1}{6}$ yards: required its surface or area.

$d=1\frac{1}{6}=\frac{7}{6}$ area $=\frac{\pi}{4}d^2$ (p. 36).

$\therefore d^2=\frac{49}{36}$ $=\cdot7854\times\frac{49}{36}$

$\pi=3\cdot1416$ $\therefore$ log. area $=$ log. $\cdot7854+$ log. $49-$ log. 36

$\therefore \frac{\pi}{4}=\cdot7854$

$$\begin{array}{lll} \text{log. } \cdot7854 & \ldots & \bar{1}\cdot895091 \\ \text{,, } 49 & \ldots & \underline{1\cdot690196} \\ & & 1\cdot585287 \\ \text{,, } 36 & \ldots & \underline{1\cdot556303} \\ \text{,, area} & \ldots & 0\cdot028984 \end{array}$$

$\therefore$ area $=1\cdot06901$ sq. yards.

149. The equatorial diameter of the earth is about 7924 miles: what is the area of the terrestrial equator?

Ans. By logarithms ....49315000 miles;
more accurately by arithmetic 49315090 ,,

Note. The Table of Logarithms in general use being only approximately correct must not be depended upon to give correct results when the natural numbers concerned exceed four or five figures: this is seen in the above example.

150. The outer and inner diameters of a flat ring are respectively $40\frac{1}{4}$ and $30\frac{1}{8}$ inches: required the area or superficial contents.

$d=40\frac{1}{4}=40\cdot25$

$d_1=30\frac{1}{8}=30\cdot125$ area $=\frac{\pi}{4}(d+d_1)\,.\,(d-d_1)$ (p. 36).

$\therefore d+d_1=70\cdot375$

$d-d_1=10\cdot125$

$\frac{\pi}{4}=\cdot7854$

$$\begin{array}{lll} \text{log. } \frac{\pi}{4} & \ldots\ldots\ldots\ldots & \bar{1}\cdot895091 \\ \text{,, } d+d_1 & \ldots\ldots\ldots & 1\cdot847419 \\ \text{,, } d-d_1 & \ldots\ldots\ldots & \underline{1\cdot005904} \\ & & 2\cdot747904 \end{array}$$

$\therefore$ surface of ring $=559\cdot633$ sq. in.

Additional examples of the use of the formulæ in page 36 *for calculating areas, surfaces, and solids.*

151. Required the area or flat surface of a circular ring whose outer and inner diameters are respectively $20\frac{1}{8}$ and $15\frac{1}{16}$ inches.

Ans. 139·908 sq. in.

152. The major and minor diameters of an ellipse are respectively $140\frac{1}{2}$ and $85\frac{3}{4}$ inches: required the area or surface.

Ans. 9462·4 sq. in.

153. What is the superficial content of an ellipse whose diameters are 625 and 562·5 yards?

Ans. 276117 sq. yards.

154. The diameter of a cylinder is $20\frac{3}{4}$ inches, and its height is 55 inches: required the convex surface and also the whole surface of solid.

Ans. Convex surface=24·9 sq. ft.
whole surface=29·6 „

155. The diameter of a cylinder is $37\frac{1}{8}$ inches, and its length $52\frac{1}{4}$ inches: required the whole surface of solid.

Ans. 57·35 sq. ft.

156. What is the whole surface of a cone whose slant side is 20 feet and the diameter of the base 2·8647 feet?

Ans. 96·44 ft.

157. What is the whole surface of a cone whose slant side is 20 feet and the circumference of the base 9 feet?

Ans. 96·44 ft.

158. What is the surface of a sphere whose diameter is 24 feet?

Ans. 1809·56 sq. ft.

159. The equatorial diameter of the earth is about 7924 miles; what is the superficial content of a sphere of that magnitude?

Ans. By logarithms ...... 197260000 miles;
more accurately by arithmetic 197260360 „

160. The length of a beam is $25\frac{1}{2}$ feet, breadth 2 feet 6 inches, and depth 1 foot 8 inches: required its cubic contents.

$a=25\frac{1}{2}=25{\cdot}5$

$b=2\frac{1}{2}=2{\cdot}5$

$c=1\frac{8}{12}=1\frac{2}{3}=\frac{5}{3}$

$V=abc=25{\cdot}5\times 2{\cdot}5\times\frac{5}{3}$

log. 25·5 ... 1·406540
„ 2·5 ... 0·397940
„ 5 ... 0·698970
2·503450
„ 3 ... 0·477121
„ V ... 2·026329

∴ V=106·25 cubic feet.

161. A beam 40 feet long is 10 inches square at the end: required its cubic contents.

Ans. $27\frac{7}{9}$ cubic ft.

162. The cubic contents of a piece of square timber is 106·25 feet, the breadth $2\frac{1}{2}$ feet, and depth $1\frac{2}{3}$ feet: find the length.

Ans. 25·5 ft.

163. The surface of a pond measures 10 acres: what is the weight of the water withdrawn when the surface falls $1\frac{1}{2}$ inches by evaporation, supposing 1 cubic foot of water to weigh 1000 ounces?*

Ans. 1519 tons nearly.

164. What is the capacity of a cylinder whose height and the circumference of its base are each 20 feet?

Ans. 636·618 cubic ft.

165. Three cubic feet are to be cut off a cylinder 44 inches in circumference: what distance from the end must the section be made?

Ans. 33·64 in.

---

* A cubic foot of pure water at 39° F. (when its density is the greatest) weighs 998·8 ounces. An imperial gallon contains 277·274 cubic inches. The imperial pound avoirdupoise is defined to be the weight of one-tenth of an imperial gallon of distilled water at the temperature of 62° F. and when the barometer stands at 30 inches: *consequently the weight* of an imperial gallon is 10 lbs.

166. The spire of a church of a conical form measures 37·6992 feet round its base; its perpendicular height is 100 feet: required its cubic contents.

Ans. 3770 cubic ft.

167. Required the cubic contents of a cone whose diameter is 7924 miles and perpendicular height 3962 miles.

Ans. by logarithms, 65128600000 miles;
more correctly by arithmetic, 65128795619 „

168. Required the volume of a globe whose diameter is 30 feet. *Ans.* 14137·2 cubic ft.

169. Required the volume of a sphere whose diameter is 7957$\frac{3}{4}$ miles.

Ans. by logarithms, 263857900000 miles;
more correctly by arithmetic, 263857437760 „

170. The axes, or diameters, of a prolate spheroid are 40 and 50: required its volume. *Ans.* 41888.

Note. A prolate spheroid is a solid generated by the revolution of an ellipse about its major diameter. An oblate spheroid is a solid formed by the revolution of an ellipse about its minor diameter.

171. What is the solidity of an oblate spheroid whose axes are 20$\frac{1}{4}$ and 10$\frac{1}{8}$? *Ans.* 2173·9.

172. The earth is an oblate spheroid whose equatorial and polar diameters are respectively 7924 and 7900 miles: required its volume.

Ans. by logarithms, 259726000000 miles;
more correctly by arithmetic, 259726141037 „

CHAPTER II.

RULES IN PLANE TRIGONOMETRY.

RULE I.—FIRST METHOD, USING HAVERSINES.*

Three sides of a plane triangle being given, to find an angle.

Put down the two sides containing the required angle, and take the difference, under which put the third side; take the sum and difference, and also the half sum and half difference.

To the arithmetical complements† of the logarithms of the first two terms in this form add the logarithms of the last two, and reject 10 in the index; the result will be the log. haversine of the required angle, which find in the table.

This rule is derived from formula B (Part II.)

$$bc \text{ hav. A} = \tfrac{1}{2}\left(a + \overline{b \sim c}\right) . \tfrac{1}{2}\left(a - \overline{b \sim c}\right).$$

The mathematical student will no doubt in some cases prefer to work out the examples by the aid of a formula rather than by the rule deduced from it. The formulæ for solving triangles will be found investigated in Part II.

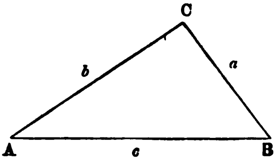
---

* If the student have no table of log. haversines, the angle may be found by the second method (p. 45).

† The arithmetical complement of a logarithm is *the difference between the logarithm and* 10; thus 10 — log. 3 is the arithmetical complement of log. 3. In practice it is most easily found by taking each figure of the logarithm from 9, except the last, and that from 10; thus, ar. co. of 2·714152 is 7·285848; ar. co. $\bar{1}$·314150 is 10·685850 (the last figure in this Example being 5, the cipher at the end not *being of any value in* that position).

EXAMPLE.

In the plane triangle* ABC, given the side $a=20$, $b=30$, and $c=40$; required angle A.



30	ar. co. log.	8·522879
40	ar. co. log.	8·397940
10	log. 15	1·176091
20	log. 5	0·698970
30	log. hav. A	8·795880
10	∴ angle A=28° 57′ 15″	
15		
5		

RULE I.—SECOND METHOD, WITHOUT HAVERSINES.

Three sides of a plane triangle being given, to find an angle.

Put down the two sides containing the required angle, and take the difference, under which put the third side; take the sum and difference, and also the half sum and half difference.

To the arithmetical complements of the logarithms of the first two terms in this form, add the logarithms of the last two, divide by 2, and look out the result as a log. sine: the angle corresponding to which will be half the required angle.†

---

* The sides of a triangle are generally distinguished by the same letters, but small, as the angles opposite to them, as in the above figure. The diagrams in this and the following problems are not drawn to scale: the method of constructing triangles by scale and compass, although not giving such correct results as the logarithmic method used in this book, ought to be known by the student. He will find ample directions for solving triangles instrumentally in any work on practical geometry and mensuration.

† This rule is derived from formula C (Part II.), viz.

$$bc\,\sin^2\frac{A}{2}=\tfrac{1}{2}(a+\overline{b\sim c})\,.\,\tfrac{1}{2}(a-\overline{b\sim c}).$$

EXAMPLES.

[1]. In the triangle ABC, given $a=512$, $b=627$, $c=430$: required C.

512	...... ar. co. log.	7·290730
627	...... ar. co. log.	7·202732
115	log. 272·5	2·435366
430	log. 157·5	2·197281
545		2)19·126109
315	log. sin.$\frac{1}{2}$C	9·563054
272·5	∴ $\frac{1}{2}$C	21° 26′ 45″
157·5		2
	and C=	42 53 30

[2]. In the triangle ABC, given $a=\cdot025$, $b=\frac{1}{8}$ or ·125, and $c=\cdot115$; required B.

·025	...... ar. co. log.	11·602060
·115	...... ar. co. log.	10·939302
·090	log. ·1075	$\bar{1}$·031408
·125	log. ·0175	$\bar{2}$·243038
·215		2)19·815808
·035	log. sin.$\frac{1}{2}$B	9·907904
·1075	∴ $\frac{1}{2}$B=	53° 59′ 30″
·0175		2
	and B=	107 59 0

In the above examples the angles have been taken out of the tables by inspection, namely, to the nearest 15″. If greater accuracy be required, the student is referred to the rules for proportioning for seconds which accompany his tables of logarithms.

EXAMPLES.

Required the angles of any plane triangle ABC whose three sides, a, b, c, are given, namely:

173.	$a=798$	*Ans.* A=	89° 45′ 37″
	$b=460$	B=	35 12 7
	$c=654$	C=	55 2 16

174.	$a=512$	*Ans.*	A=	54°	8′	11″
	$b=627$		B=	82	58	11
	$c=430$		C=	42	53	38
175.	$a=649$	„	A=	56	4	6
	$b=586$		B=	48	31	4
	$c=757$		C=	75	24	50
176.	$a=627$	„	A=	29	44	2
	$b=1140$		B=	115	36	32
	$c=718\cdot9$		C=	34	39	26
177.	$a=\cdot025$	„	A=	10	58	0
	$b=\frac{1}{8}$		B=	107	58	50
	$c=\cdot115$		C=	61	3	10
178.	$a=\cdot8$	„	A=	107	46	6
	$b=\cdot672$		B=	53	7	24
	$c=\cdot275$		C=	19	6	30
179.	$a=\frac{1}{2}$	„	A=	28	14	14
	$b=\frac{3}{4}$		B=	45	12	34
	$c=1\cdot013$		C=	106	33	12
180.	$a=\frac{1}{4}$	„	A=	20	11	24
	$b=\cdot541$		B=	48	19	11
	$c=\cdot674$		C=	111	29	23

In the above examples the angles have been computed to the nearest second; a tedious operation, which is seldom required: the result correct to the nearest minute being, in most cases, considered sufficient.

RULE II.

Of two sides and two opposite angles, any three being given, to find the fourth.

Write down a proportion, having for the first two terms the two sides concerned, and for the third term the sine of the angle opposite to the first side put down, and for the fourth term the side of the angle opposite the side in the second term of the proportion; mark the term required with a stroke *of the pen* underneath.

If the term marked be a *middle* term: add the logarithms of the two extreme terms, and subtract the logarithm of the middle term not marked.

If the term marked be an *extreme* term: add the logarithms of the two middle terms, and subtract the logarithm of the extreme term.*

The result will be the logarithm of the required term. See examples to Art. (21), p. 16.

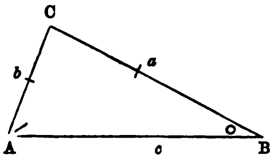
Note [1]. When two angles of a plane triangle are known, the third can be found by adding together the two known angles and subtracting the result from 180°; the remainder will be the third angle: hence, if two angles and the adjacent side be given, the third angle is also known, and thence by the above rule the other two sides can be found.

[2]. This rule depends on the property of triangles, that the sides are to one another in the same proportion as the *sines* of an angle opposite to them; and it will be in general the easiest way of working, to write down such proportion, and find the unknown term as in Art. (21).

EXAMPLES.

In a plane triangle ABC, given $a=50\frac{1}{4}$, $b=25$, and $A=68° 48'$: to find the other parts.

Draw the triangle ABC, making the given parts as near the truth as possible without using instruments: mark the given parts (A, a, b) of the triangle with a stroke of the pen, as in the figure: then, since the rule requires that the four parts concerned should be two sides and the two angles opposite to them, the part to be found must be B; mark this with a different character (these marks will render it more easy to apply the rule); then write



---

* Derived from formula F (Part II.), namely,

$$\frac{a}{b}=\frac{\sin. A}{\sin. B}, \text{ or } a : b :: \sin. A : \sin. B.$$

down a proportion as directed by the rule, and proceed to find B.

The angles A and B being known, C is easily found Note [1]; and thence, by a proportion, the side *c*.

Calculation.

[1]. To find B.

a : *b* : : sin. A : sin. B

log. *b* ……	1·397940
log. sin. A	9·969567
	11·367507
log. *a* ……	1·701136
log. sin. B	9·666371

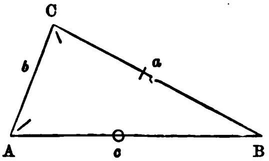
B=27° 38′ 15″

[2]. To find C.

B ……	27°	38′	15″
A ……	68	48	0
	96	26	15
	180		
	C=83°	33′	45″

[3]. To find the side *c*.

Draw the triangle ABC again, but mark the given parts A, *a*, and C with a stroke of the pen, and the required part *c* with a different mark, as in the figure; then make a proportion of the four parts marked; thus



c : *a* : : sin. C : sin. A

log. *a* ………………	1·701136
log. sin. C …………	9·997253
	11·698389
log. sin. A …………	9·969567
log. *c* ………………	1·728822

.·. *c*=53·56

EXAMPLES.

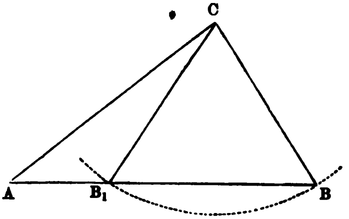
Find the other parts of the triangle ABC, having given:

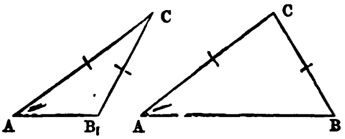
181.	*a*=214	*Ans.* B= 36° 6′ 30″
	b=191	C=102 34 15
	A=41° 19′ 15″	c=316·3

182.	$a=17{\cdot}25$	*Ans.*	$C=27^\circ\ 7'\ 15''$
	$c=10\frac{3}{4}$ or $10{\cdot}75$		$B=105\ \ 52\ \ 15$
	$A=47^\circ\ 0'\ 30''$		$b=22{\cdot}69$
183.	$a=96$	„	$b=73{\cdot}98$
	$c=48$		$B=49^\circ\ 0'$
	$A=101^\circ\ 41'$		$C=29\ \ 19$
184.	$c=376$	„	$C=91\ \ 43$
	$A=48^\circ\ 3'$		$a=279{\cdot}7$
	$B=40^\circ\ 14$		$b=243{\cdot}0$
185.	$A=60^\circ$	„	$C=48^\circ\ 0'$
	$B=72^\circ$		$c=207{\cdot}6$
	$a=242$		$b=265{\cdot}76$
186.	$a=2\frac{3}{4}$	„	$C=88^\circ\ 2'\ 40''$
	$A=43^\circ\ 24'\ 10''$		$b=3$
	$B=48\ \ 33\ \ 10$		$c=4$
187.	$b=\frac{3}{4}$	„	$A=28^\circ\ 13'$
	$a=\frac{1}{2}$		$C=106\ \ 37$
	$B=45^\circ\ 10'$		$c=1{\cdot}013$
188.	$A=20\ \ 10$	„	$C=111^\circ\ 35'$
	$B=48\ \ 15$		$b={\cdot}541$
	$a=\frac{1}{4}$		$c={\cdot}6743$
189.	$a={\cdot}02$	„	$B=90^\circ$
	$A=11^\circ\ 32'\ 15''$		$c={\cdot}09797$
	$C=78\ \ 27\ \ 45$		$b={\cdot}1$
190.	$A=22\ \ 20\ \ 0$	„	$C=108^\circ\ 12'\ 30''$
	$B=49\ \ 27\ \ 30$		$a=1$
	$c=2\frac{1}{2}$		$b=2$

The Ambiguous Case.

(36). When two sides and the angle opposite the *less* side are given, each of the quantities sought will have two distinct values. For suppose that, in the triangle ABC, the sides BC and AC and the angle A, opposite to the *less* side BC, are given; then, if BC be not perpendicular to AB, from C as a centre, with radius CB, describe an arc cutting AB in another point B_1, and join CB_1; it is manifest that there will be *two* triangles, ACB and ACB_1, having the sides AC, CB, and angle A, in one triangle, and the sides AC, CB_1, and angle A, in the other, of the same magnitude, while the remaining parts are different. Hence it is evident that when such parts of a plane triangle are given, the results will admit of two different values. This may also be shown by considering that the angle ABC or AB_1C is found by the formula $a : b :: \sin. A : \sin. B$, or $a : b :: \sin. A : \sin. AB_1C$; therefore the sines of the angles B and AB_1C have the same numerical value (each being $=\frac{b . \sin. A}{a}$): they are consequently the *supplements* of each other. This also appears from the above figure; for since $CB=CB_1$, the angles B and CB_1B are equal; therefore AB_1C, the supplement of CB_1B, is also the supplement of B. To find these two results we must, therefore, proceed as follows: Having found the acute angle B by the rule, subtract it from 180° for the angle AB_1C in the other triangle; the remaining angles ACB, ACB_1, and sides AB, AB_1, may then be found in the usual manner.





EXAMPLE.

In the triangle ABC, or AB_1C, given $a=232$, $b=345$, and $A=37° 20'$; to find the other parts.

Calculation of angle B.

$a : b :: \sin. A : \sin. B$

log. b	2·537819
log. sin. A	9·782796
	12·320615
log. a	2·365488
log. sin. B	9·955127

First solution, B=64° 24′

There are two solutions.

In the first,

$B=64° 24'$

in the second,

$B_1=115° 36'$

Second solution, $B_1=115° 36'$

Calculation of angle C.

$$\begin{array}{rrr} C = 180^\circ - A - B & & \\ A = & 37^\circ & 20' \\ B = & 64 & 24 \\ \hline & 101 & 44 \\ & 180 & \\ \hline \therefore C = & 78 & 16 \end{array}$$

Calculation of side c.

$c : a :: \sin. C : \sin. A$

$$\begin{array}{lr} \log. a \text{ .... } & 2 \cdot 365488 \\ \log. \sin. C \text{ ...... } & 9 \cdot 990829 \\ \hline & 12 \cdot 356317 \\ \log. \sin. A \text{ ...... } & 9 \cdot 782796 \\ \hline \log. c \text{ .... } & 2 \cdot 573521 \end{array}$$

$\therefore c = 374 \cdot 6$

Calculation of angle C.

$$\begin{array}{rrr} C = 180^\circ - A - B_1 & & \\ A = & 37^\circ & 20' \\ B_1 = & 115 & 36 \\ \hline & 152 & 56 \\ & 180 & \\ \hline \therefore C = & 27 & 4 \end{array}$$

Calculation of side c.

$c : a :: \sin. C : \sin. A$

$$\begin{array}{lr} \log. a \text{ .... } & 2 \cdot 365488 \\ \log. \sin. C \text{ ...... } & 9 \cdot 658037 \\ \hline & 12 \cdot 023525 \\ \log. \sin. A \text{ ...... } & 9 \cdot 782796 \\ \hline \log. c \text{ .... } & 2 \cdot 240729 \end{array}$$

$\therefore c = 174 \cdot 07$

EXAMPLES.

Find the other parts of the triangle ABC, having given :

191. $a = 178 \cdot 3$ *Ans.* $A = 54^\circ\ 2'$, or $125^\circ\ 58'$
$b = 145$ $C = 84\ \ 48$, or $12\ \ 52$
$B = 41^\circ\ 10'$ $c = 219 \cdot 32$, or $49 \cdot 05$

192. $a = 2597 \cdot 84$ „ $B = 80^\circ\ 39'\ 45''$, or $99^\circ\ 20'\ 15''$
$b = 3084 \cdot 33$ $C = 43\ \ 7\ \ 30$, or $24\ \ 27$
$A = 56^\circ\ 12'\ 45''$ $c = 2136 \cdot 7$, or $1293 \cdot 7$

Rule III.

Two sides and the included angle being given, to find the remaining angles.

Subtract the given angle from 180°, and thus find the supplement : divide the supplement by 2. Put down the two given sides, and take their sum and difference.

To the log. tangent of half the supplement of the given angle, add the log. of the difference of the given sides ; and from the result subtract the log. of the sum of the given sides; the remainder will be the log. tangent of half the difference of the required angles: which take from the tables.

To half the supplement of the given angle, *add* half the difference of required angles just found, and the sum will be the *greater* of the two angles required. To find the *less* of the two required angles, *subtract* the half difference.

In this case it is easier to work by the formula rather than by the above rule deduced from it. Thus, if a, b, and C be the given quantities, and the remaining angles A and B are required: by formula G (Part II.) we have

$$\frac{a+b}{a-b}=\frac{\tan.\frac{1}{2}(A+B)}{\tan.\frac{1}{2}(A-B)},$$

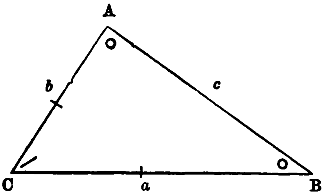
or expressed as a proportion:

$$a+b : a-b :: \tan.\ \tfrac{1}{2}(A+B) : \tan.\ \tfrac{1}{2}(A-B).$$

The first three terms in this proportion are known; for half the sum of the required angles, or $\frac{1}{2}(A+B)=\frac{1}{2}(180°-C)$ (the half supplement of given angle): hence the fourth term is easily found, which gives us half the difference of the required angles. The *sum* of the half sum and half difference of the required angles will be the angle opposite the *greater* of the two given sides; and their *difference* will be the *less* angle.

EXAMPLES.

Given $a=798$, $b=460$, and $C=55°\ 2'\ 15''$: required the angles A and B.



$$a+b : a-b :: \tan.\ \tfrac{1}{2}(A+B) : \underline{\tan.\ \tfrac{1}{2}(A-B)}$$

a ...	798		180° 0′ 0″
b ...	460		55 2 15 ... C
$a+b$ ...	1258	2)	124 57 45 ... A+B
$a-b$ ...	*338*		62 28 52 ... $\frac{1}{2}$(A+B)

log. tan. ½ (A+B)	10·283138	∴ ½ A−½ B=	27° 16′ 45″
log. $(a-b)$ ...	2·528917	and ½ A+½ B=	62 28 52
	12·812055	∴ A=	89 45 37
log. $(a+b)$ ...	3·099681	and B=	35 12 7
log. tan. ½ (A−B)	9·712374		

Find the other two angles of the triangle ABC, having given:

193.	$a=399$ $b=230$ C=55° 2′ 15″	*Ans.*	A=89° 45′ 37″ B=35 12 7
194.	$b=64$ $c=70$ A=66° 20′ 30″	„	B=52 54 30 C=60 45 0
195.	$a=512$ $b=627$ C=42° 53′ 38″	„	A=54 8 11 B=82 58 11

RULE IV.—FIRST METHOD, USING HAVERSINES.*

Given two sides and the included angle, to find the third side.

Add together 10·602060, log. haversine of given angle, and logarithm of each given side: take half the sum, from which subtract log. of difference of the given sides. Look in the tables for the remainder as a log. tangent, and take out the corresponding log. sine, which subtract from the half sum just used. The remainder will be the log. of the required side.

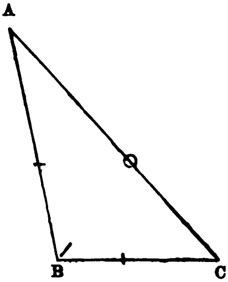
Note. When the half sum is first found, write it down again a little to the right hand, for the convenience of placing under it the log. sine to be subtracted.*

---

* If the student have no table of haversines, he may work the examples in this rule by finding, in the first place, the two other angles by Rule III., and then the required side by Rule II.; or he may proceed as follows: Instead of using log. haversine, as directed in the rule, take out twice the log. sine of half the given angle, and reject the 10 in the index of the constant log., then proceed as in the above *rule; or he may find* the side required by the next method.

EXAMPLES.

In the triangle ABC, given $a=20$, $c=30$, and $B=100°$: required b.



	constant log. .	10·602060		
	log. hav. B . .	9·768508		
a . . 20	. log. a	1·301030		
c . . 30	. log. c	1·477121		
$a \sim c$ . . 10		2)23·148719		
		11·574359	. . . .	11·574359
	log. $(a \sim c)$ . .	1·000000		
	log. tan.	10·574359	log. sin. .	9·985104
			log. b . . .	1·589255
				$b=38·84$

RULE IV.—SECOND METHOD, WITHOUT HAVERSINES.

Given two sides and the included angle, to find the third side.

Add together the two given sides, and also subtract them: thus find their sum and difference. Put down also half the given angle.

Under heads (1) and (2) write down the following logarithms.

Under (1) and (2) the log. sum of given sides.

Under (1) the log. difference of given sides.

Take the difference of the logs. under (1). Again, under (1) put the log. tangent of half the given angle; under (2) put the log. sine of half the given angle.

Add together the two logarithms under (2), and also the last two logarithms under (1).

Look out the result under (1) as a log. tangent, and take

---

* By formula H (Part II.)

$$a=\frac{\sqrt{4\ bc \text{ hav. A}}}{\sin.\ \theta} \text{ where } \tan\theta=\frac{\sqrt{4\ bc \text{ hav. A}}}{b \sim c}$$

out the log. sine corresponding thereto, which subtract from the log. under (2). The result will be log. of required side.*

EXAMPLE.

In the plane triangle ABC, given $a=178$, $b=145$, and $C=84^\circ\ 48'$: required the third side c.

	[1].		[2].	
a .. 178	log. $(a+b)$ ..	2·509203	log. $(a+b)$.	2·509203
b .. 145	log. $(a-b)$ ..	1·518514	log. sin. $\frac{1}{2}$C .	9·828855
$a+b$ .. 323	difference ...	0·990689	sum	12·338058
$a-b$ .. 33	log. tan. $\frac{1}{2}$C .	9·960530	log. sin. arc .	9·997299
$\frac{1}{2}C=42^\circ 24'$	log. tan. arc .	10·951219	log. c	2·340759
			∴ side $c=219{\cdot}2$	

Find the third side in the following examples:

196. $a=798$
$b=460$
$C=55^\circ\ 2'\ 16''$ *Ans.* $c=654$

197. $c=48$
$a=96$
$B=49^\circ\ 0'$ „ $b=73{\cdot}98$

198. $a=512$
$b=627$
$C=42^\circ\ 53'\ 38''$ „ $c=430$

199. $b=2$
$c=2\frac{1}{2}$
$A=22^\circ\ 20'\ 0''$ „ $a=1$

RIGHT-ANGLED TRIANGLES.

Right-angled triangles may be solved by the rules already given; but the easiest way is to make use of the trigonometrical ratios or definitions for the sine, tangent, &c. given in Art. (5), p. 3, as in the following rule.

---

* This rule is deduced from a formula investigated at p. 91 (Part II.), namely,

$$a=\frac{(b+c)\ \text{sin.}\ \frac{1}{2}A}{\text{sin.}\ \theta}\quad\text{where}\ \tan\theta=\frac{\overline{b+c}\ \tan\frac{1}{2}A}{b\sim c}$$

RULE V.

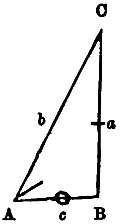
Right-angled Plane Triangles.

Select that expression or ratio in p. 3 which contains the two known terms and also the term required, and when possible let the numerator of the ratio or fraction selected be the part required. Reduce it to tabular logarithms, by Art. (31), p. 25, and then find the value of the unknown term in the usual manner. See Ex. 117, p. 27.

EXAMPLES.

1. In the right-angled plane triangle ABC, given $B=90°$, $a=42$, and $A=50° 10'$: required the other parts.

Mark the given parts, namely A and a, with a stroke of the pen; and if c is the part about to be found, mark it with a different character, as in the figure. Then



[1]. To find c.

By p. 3, ratio (d), $\frac{c}{a}=\text{cot. } A$, or $c=a \text{ cot. } A$

$$\therefore \text{log. } c=\text{log. } a+\text{log. cot. } A-10$$

Calculation of c.

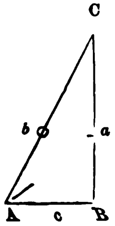
$a=42$	log. a	1·623249
$A=50° 10'$	log. cot. A ...	9·921247
	log. c	1·544496

$\therefore c=35·03$

[2]. To find b.

Draw the triangle again; mark the given parts A and a with a stroke, and b, the part required, with a different mark as before: then by ratio (b), p. 3, $\frac{b}{a}=\text{cosec. } A$, or $b=a \text{ cosec. } A$

$$\therefore \text{log. } b=\text{log. } a+\text{log. cosec. } A-10$$



$a=42$	log. a	$1\cdot623249$
$A=50^\circ\ 10'$	log. cosec. A	$\underline{10\cdot114689}$
	log. b	$1\cdot737938$

$\therefore b=54\cdot7$

To find C : $C=90^\circ-A=39^\circ\ 50'$

2. Given $a=\cdot02$, $c=0\cdot1$, and $B=90^\circ$: required the other parts. Make a triangle ABC, right-angled at B (see last fig.); mark the given sides a and c with a stroke of the pen, mark also the part required with a different character.

Let the part required be A.

[1]. To find A.

By p. 3, ratio (c), tan. $A=\dfrac{a}{c}$

$\therefore$ log. tan. A$-10=$log. $a-$log. c

or log. tan. A$=10+$log. $a-$log. c

Calculation.

$a=\cdot02$	log. a	$\bar{2}\cdot301030$
$c=\cdot1$		$\underline{10}$
		$8\cdot301030$
	log. c	$\underline{\bar{1}\cdot000000}$
	log. tan. A ...	$9\cdot301030$

$\therefore A=11^\circ\ 18'\ 30''$

[2.] To find side b.

By p. 3, ratio (e), $\dfrac{b}{c}=$sec. A

$\therefore b=c$ sec. A

log. $b=$log. $c+$log. sec. A-10

$c=\cdot1$	log. c	$\bar{1}\cdot000000$
$A=11^\circ\ 18'\ 30''$	log. sec. A ...	$\underline{10\cdot008514}$
		$9\cdot008514$
		$\underline{10}$
	log. b	$\bar{1}\cdot008514$

$\therefore b=\cdot102$

To find C : $C=90^\circ-A=78^\circ\ 41'\ 30''$

When two sides of a right-angled triangle are given, the third may be found by the well-known property of right-angled plane triangles, namely, ‘the square of the side opposite the right angle is equal to the sum of the squares of the sides containing the right angle’ (Euc. i. 47): thus, in a triangle ABC, right-angled at C, $a^2+b^2=c^2$; and if $a=\cdot 02$, and $b=\cdot 1$, we have

$$c^2=(\cdot 02)^2+(\cdot 1)^2, \text{ or } c=\sqrt{\cdot 0104}=\cdot 102 \text{ nearly.}$$

If one of the given sides is the hypothenuse, this method of finding the third side may be simplified: thus, in the above example, given $c=\cdot 102$, and $a=\cdot 02$; to find b.

$$\text{Since } a^2+b^2=c^2$$

$$\therefore b^2=c^2-a^2=(c+a).(c-a)=\cdot 122\times\cdot 082=\cdot 010004$$

$$\therefore b=\cdot 1 \text{ nearly.}$$

Find the other parts of the right-angled triangle ABC, having given

200.	$A=52^\circ\ 38'\ 0''$	*Ans.*	$C=37^\circ\ 22'$
	$b=45$		$a=35\cdot 76$
	$B=90^\circ$		$c=27\cdot 31$
201.	$A=49^\circ\ 14'$	„	$C=40^\circ\ 46'$
	$c=331$		$a=384$
	$B=90^\circ$		$b=506\cdot 8$
202.	$A=56^\circ\ 29'\ 15''$	„	$a=3555$
	$b=4264\cdot 3$		$c=2354$
	$B=90^\circ$		$C=33^\circ\ 30'\ 45''$
203.	$A=4^\circ\ 44'$	„	$c=8390$
	$a=694\cdot 73$		$b=8419$
	$B=90^\circ$		$C=85^\circ\ 16'$
204.	$b=\cdot 2$	„	$a=\cdot 1286$
	$A=40^\circ$		$c=\cdot 1532$
	$B=90^\circ$		$C=50^\circ$
205.	$c=\cdot 04$	„	$a=\cdot 04767$
	$C=40^\circ$		$b=\cdot 06223$
	$B=90^\circ$		$A=50^\circ$

206. $a = 1777 \cdot 5$ *Ans.* $A = 56° \; 29' \; 15''$
$c = 1177$ $C = 33 \quad 30 \quad 45$
$B = 90°$ $b = 2132 \cdot 1$

RULE VI.

Two sides and the included angle of a plane triangle being given, to find the area.

Add together the logarithms of the two given sides, and log. sine of the given angle; the sum, rejecting 10 in the index, will be the logarithm of twice the required area.*

EXAMPLE.

Given $a = 798$, $b = 460$, and $C = 55° \; 2' \; 15''$: required the area.

log. a .... 2·902003
log. b .... 2·662758
log. sin. C .... 9·913563
log. 2 area .... 5·478324
∴ 2 area 300832
and area 150416

207. Given $a = 245$ yards, $b = 760$ yards, and $C = 60°$: required the area. *Ans.* 80627 square yards.

RULE VII.

Three sides of a plane triangle being given, to find the area.

From half the sum of the three sides, subtract each side separately. Add together the log. of the half sum and the logarithms of the three remainders. Half the result will be the logarithm of the area.†

---

* By formula I (Part II.):

$$2 \text{ area} = b\,c \text{ sin. } A.$$

† By formula K (Part II.):

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$
$$\text{where } s = \tfrac{1}{2}(a+b+c).$$

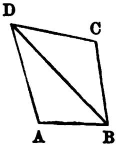
EXAMPLE.

208. Given $a=798$, $b=460$, and $c=654$: required the area.

a ... 798				log. 956 ... 2·980458
b ... 460	956	956	956	log. 158 ... 2·198657
c ... 654	798	460	654	log. 496 ... 2·695482
2)1912	158	496	302	log. 302 ... 2·480007
956				2)10·354604
			∴ area=150418	5·177302

209. In the trapezium ABCD, AB=90 yards, BC=100 yards, CD=110 yards, DA=120 yards, the diagonal BD=178·8 yards: required the area of the trapezium.

Ans. 9768·7 square yards.



SPHERICAL TRIGONOMETRY.

PART I.

CHAPTER III.

RULES IN SPHERICAL TRIGONOMETRY.

RULE VIII.—FIRST METHOD, USING HALF HAVERSINES.

Three sides of a spherical triangle being given, to find an angle.*

Put down the two sides containing the required angle, and take the difference, under which put the third side; take the sum and difference.

Add together the log. cosecants of the two first terms in this form (rejecting the tens from the index), and the half log. haversines of the two last terms.† The result will be the log. haversine of the required angle.‡

RULE VIII.—SECOND METHOD, WITHOUT HAVERSINES.

Three sides of a spherical triangle being given, to find an angle.

Put down the two sides containing the required angle,

* A *spherical* triangle is that part of the surface of a sphere which is bounded by arcs of three *great* circles, that is, three circles whose planes pass through the centre of the sphere. The three arcs are the *sides* of the triangles; and any one of its *angles* is the same as the inclination of the planes of the sides containing the angle. For other definitions in Spherical Trigonometry, see *Trigonometry*, Part II.

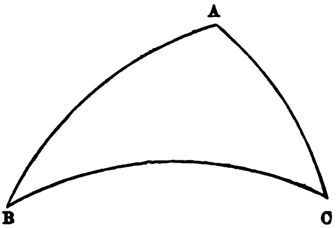
† If the student have no table of half log. haversines, he may, after a little practice, take out of the table of haversines their halves *by inspection:* or, the log. haversines may be written down on the paper, and then divided by 2. See example.

‡ By formula O (Part II.)

$$\text{hav. } A = \text{cosec. } b \,.\, \text{cosec. } c \sqrt{\text{hav. } (a+\overline{b\sim c}) \,.\, \text{hav. } (a-\overline{b\sim c})}$$

and take the difference; under which put the third side; take the sum and difference, and the half sum and half difference.

Add together the log. cosecants of the two first terms in this form (rejecting the tens from the index), and the log. sines of the two last terms; divide by 2, and look out the result as a log. sine; the angle corresponding to which will be half the required angle.*



210. In the spherical triangle ABC, given $a=124°\ 10'\ 0''$, $b=89°\ 0'\ 15''$, $c=180°\ 40'\ 0''$: required the angle A.

By First Method.

	89° 0′ 15″	log. cosec.	0·000066
	108 40 0	log. cosec.	0·023468
	19 39 45	½ log. hav. S	4·978000
	124 10 0	½ log. hav. D	4·898018
S ...	143 49 45	log. hav. A	9·899552
D ...	104 30 15		∴ A=125° 56′ 45″

Or thus (according to note † on preceding page):

log. hav. of the two last terms	9·796036 9·956000
	2)19·752036
half log. hav.	9·876018
log. cosecants two first terms	0·023468 0·000066
log. hav. A	9·899552
	∴ A=125° 56′ 45″

---

* By formula O (Part II.)

$$\sin^2\frac{A}{2}=\text{cosec. } b\,.\,\text{cosec. } c\,.\,\sin.\tfrac{1}{2}\,(a+\overline{b\sim c})\,.\,\sin.\tfrac{1}{2}\,(a-\overline{b\sim c})$$

By Second Method.

	89° 0′ 15″	log. cosec.	0·000066
	108 40 0	log. cosec.	0·023468
	19 39 45	log. sin. (½ S)	9·978000
	124 10 0	log. sin. (½ D)	9·898018
S ...	143 49 45		2)19·899552
D ...	104 30 15	log. sin. ½ A ...	9·949776
½S ...	71 54 52	∴ ½ A=	62° 58′ 15″
½D ...	52 15 7		2
		and A=	125 56 30

EXAMPLES.

Find the three angles of the spherical triangle ABC, having given:

211.	a= 49° 10′ 0″	*Ans.*	A= 59° 2′ 0″
	b= 58 25 0		B= 74 54 0
	c= 56 42 0		C= 71 18 30
212.	a=119 42 20	„	A=115 38 45
	b=108 4 18		B= 99 21 15
	c= 68 53 42		C= 75 31 30
213.	a= 87 10 15	„	A= 81 24 0
	b= 62 36 45		B= 61 31 15
	c=100 10 15		C=102 59 0

RULE IX.—FIRST METHOD, USING HAVERSINES.

Two sides and the included angle being given, to find the remaining side.

Add together 6·301030, the log. sines of the given sides, and log. haversine of given angle; reject 30 from the index of the sum, and take out the natural number of the resulting logarithm. To this natural number add the natural versine of the difference of the given sides; the sum will be the natural versine of the required side, which find from the tables.*

RULE IX.—SECOND METHOD, WITHOUT HAVERSINES.

Two sides and the included angle being given, to find the remaining side.

Add together 6·301030, the log. sines of the given sides, and twice the log. sine of half the given angle, rejecting 40 from the index of the sum; and take out the natural number of the resulting logarithm. To this natural number add the natural versine of the difference of the given sides; the sum will be the natural versine of the required side, which find from the tables.†

RULE IX.—THIRD METHOD (REQUIRING ONLY THE COMMON TABLE OF SINES, &c.).

Given two sides and the included angle, to find the third side.

Take the difference between the two given sides: divide it by 2, and thus get half the difference of given sides: find also half the given angle.

Add together the log. sines of the two given sides, and divide the result by 2, to which add the log. sine of half the given angle. Call this sum log. M. From log. M subtract log. sine of half the difference of given sides, and look in the tables for this result as a log. tangent; and take out the corresponding log. sine, which subtract from log. M. The remainder will be the log. sine of half the required side.‡

---

* By formula, p. 94, Part II., we have

$$\text{vers. } a = \text{vers. } (b \sim c) + 2 \text{ sin. } b \text{ . sin. } c \text{ . hav. A.}$$

† Also, at same page,

$$\text{vers. } a = \text{vers. } (b \sim c) + 2 \text{ sin. } b \text{ . sin. } c \text{ . sin.}^2 \frac{A}{2}$$

‡ By formula, p. 95, Part II.,

$$\text{sin. } \frac{a}{2} = \frac{\text{sin. } \tfrac{1}{2} A \sqrt{\text{sin. } b \text{ . sin. } c}}{\text{sin } \theta}$$

$$\text{where tan. } \theta = \frac{\text{sin. } \tfrac{1}{2} A \sqrt{\text{sin. } b \text{ . sin. } c}}{\text{sin. } \tfrac{1}{2} (b \sim c)}$$

Note. When log. M is first found, write it down again a little to the right hand, for the convenience of placing under it the log. sine to be subtracted.

EXAMPLE.

In the spherical triangle ABC, given c=119° 42′ 20″, b=108° 4′ 18″, and A=75° 31′ 30″: required the remaining side a.

By First Method.

				const. log. ...	6·301030
c=	119°	42′	20″	... log. sin. c ...	9·938818
b=	108	4	18	... „ sin. b ...	9·978031
$c-b$=	11	38	2	„ hav. A ..	9·574056
				log. ...	5·791935

nat. No. 619349

nat. ver. ($c-b$) ... 20544

nat. ver. a 639893

$\therefore$ a=68° 53′ 36″

By Second Method.

				const. log. ...	6·301030
c=	119°	42′	20″	... log. sin. c ...	9·938818
b=	108	4	18	... „ sin. b ...	9·978031
$c-b$=	11	38	2	„ sin. $\frac{1}{2}$ A.	9·787028
$\frac{1}{2}$A=	37	45	45	„ sin. $\frac{1}{2}$ A.	9·787028
				log. ...	5·791935

nat. No. 619349

nat. ver. ($c-b$) ... 20544

nat. ver. a 639893

$\therefore$ a=68° 53′ 36″

By Third Method.

$b=108^\circ\ 4'\ 18''$ log. sin. b ... 9·978031
$c=119\ 42\ 20$ log. sin. c ... 9·938818
$2)\overline{11\ 38\ 2}$ $2)\overline{19{\cdot}916849}$
$\frac{1}{2}(b\sim c)\ 5\ 49\ 1$ 9·958424
log. sin. $\frac{1}{2}$ A ... 9·787028
$\frac{1}{2}\mathrm{A}=37\ 45\ 45$ log. M. ......19·745452 ....19·745452
log. sin. $\frac{1}{2}(b\sim c)$... 9·005805
log. tan. ......10·739647. log. sin. 9·992916
log. sin. $\frac{1}{2}a$ ... 9·752536
$\therefore\ \frac{1}{2}a=34^\circ\ 26'\ 45''$
2
$\therefore$ side $a=68\ 53\ 30$

EXAMPLES.

Find the third side of the spherical triangle ABC, having given :

214.	A= 96° 32′ 0″	
	b= 76 42 0	
	c= 89 10 30	*Ans.* a= 96° 9′ 57″
215.	A= 50 0 0	
	b= 70 45 10	
	c= 62 10 15	„ a= 46 19 32
216.	a=100 8 42	
	b= 98 10 5	
	C= 88 24 24	„ c= 87 0 50
217.	b=118 2 14	
	c=120 18 33	
	A= 27 22 34	„ a= 23 57 9
218.	a= 87 10 15	
	b= 62 36 45	
	C=102 58 30	„ c=100 9 38
219.	a= 69 19 10	
	b= 78 59 14	
	C=110 48 42	„ c=104 59 57

RULE X.

Of two sides and the opposite angles, any three being given, to find the fourth.

Write down a proportion having for the two first terms the sines of the two sides concerned, and for the third term the sine of the angle opposite to the first side put down, and for the fourth term the sine of the angle opposite to the other side put down in the proportion. Mark the term required, and proceed as in the corresponding rule for plane triangles (p. 47).*

EXAMPLES.

[1]. In the spherical triangle ABC, given $a=70° 10' 30''$, $b=80° 5'$, and $B=33° 15'$: required the angle A.

[2]. $b=119° 42' 20''$, $a=108° 4' 18''$, and $A=99° 21' 30''$: required B.

sin. a : sin. b : : $\underline{\text{sin. A}}$: sin. B

log. sin. a ... 9·973466
„ sin. B ... 9·739013
19·712479
„ sin. b ... 9·993462
„ sin. A ... 9·719017
∴ $A=31° 34' 30''$

sin. a : sin. b : : sin. A : $\underline{\text{sin. B}}$

log. sin. b ... 9·938818
„ sin. A ... 9·994181
19·932999
„ sin. a ... 9·978031
„ sin. B ... 9·954968
∴ $B=180°-64° 21' 30''$
$=115° 38' 30''$

In example [1], a being less than b, A is less than B, and therefore A is less than 90°.

In example [2], b being greater than a, B is greater than A, and therefore B is greater than 90°: hence the angle taken out of the tables must be subtracted from 180° to get B.

Unless by some limitation of this kind it should be determined that the required angle is less or greater than 90°,

---

* By formula N (Part II.)

$$\frac{\sin. a}{\sin. b}=\frac{\sin. A}{\sin. B}$$

this rule is considered ambiguous. See corresponding rule for plane triangles (p. 50).

RULE XI.

Two sides and the included angle being given, to find the other two angles.

Take the sum and difference of the two given sides, and the half sum and half difference, and also half the given angle.

Under heads [1] and [2], put down the following logarithms:

Under [1] and [2], log. cot. of half the given angle.

$$\left.\begin{array}{l}\text{Under [1], log. cosec.}\\ \text{Under [2], log. sec.}\end{array}\right\}\begin{array}{l}\text{of half the sum of the given sides}\\ \text{(rejecting 10 in each index).}\end{array}$$

$$\left.\begin{array}{l}\text{Under [1], log. sin.}\\ \text{Under [2], log. cos.}\end{array}\right\}\text{of half the difference of given sides.}$$

Add together the logarithms under [1] and [2], and (rejecting 10 from each index) the results will be the log. tangents of half the difference and half the sum of the required angles respectively, which take from the tables (*a*).

The sum of the two arcs thus found will be the angle opposite to the greater of the given sides; and their difference will be the angle opposite to the less of the given sides.*

(*a*). *Note.* If half the sum of the given sides be greater than 90°, half the sum of the required angles will also be greater than 90°: in this case, the arc found as above under [2] must be subtracted from 180° to get half the sum of the required angles.

EXAMPLE.

220. In the spherical triangle ABC, given $a = 124° \, 10'$,

---

* By formula T (Part II.)

$$\tan. \tfrac{1}{2}\,(A+B) = \frac{\cos. \tfrac{1}{2}\,(a-b)}{\cos. \tfrac{1}{2}\,(a+b)} \,.\, \cot. \tfrac{1}{2}\,C$$

$$\tan. \tfrac{1}{2}\,(A-B) = \frac{\sin. \tfrac{1}{2}\,(a-b)}{\sin. \tfrac{1}{2}\,(a+b)} \,.\, \cot. \tfrac{1}{2}\,C$$

$b=89° 0' 15''$, $C=112° 1' 30''$: required the other two angles, A and B.

$a=$	124°	10′	0″		$C=$	112°	1′	30″
$b=$	89	0	15		$\therefore \frac{1}{2}C=$	56	0	45
$a+b=$	213	10	15	....	$\frac{1}{2}(a+b)=$	106	35	7
$a-b=$	35	9	45	....	$\frac{1}{2}(a-b)=$	17	34	52

[1].		[2].	
cot. $\frac{1}{2}C$	9·828783	cot. $\frac{1}{2}C$	9·828783
cosec. $\frac{1}{2}(a+b)$ ...	0·018455	sec. $\frac{1}{2}(a+b)$ ...	0·544480
sin. $\frac{1}{2}(a-b)$	9·480090	cos. $\frac{1}{2}(a-b)$ ...	9·979225
tan. $\frac{1}{2}(A-B)$ ...	9·327328	tan. arc	10·352488
$\therefore \frac{1}{2}(A-B)$ ...	11° 59′ 45″	$\therefore$ arc ...	66° 3′ 0″*
			180
		$\therefore \frac{1}{2}(A+B)$ ...	113 57 0
		and $\frac{1}{2}(A-B)$ ...	11 59 45
		$\therefore A=$	125 56 45
		$B=$	101 57 15

221. Given $b=89° 0' 15''$, $c=108° 40'$, $A=125° 56' 45''$: to find the angles B and C.

Ans. $B=101° 57' 15''$, $C=112° 2'$.

Examples to this and the following rule are easily formed out of those already given; as Ex. 211, &c.

Rule XII.

Given two angles and the included side, to find the other two sides.

Take the sum and difference of the two given angles, and the half sum and half difference, and also half the given side.

Under heads [1] and [2] put down the following logarithms:

Under [1] and [2], log. tan. of half the given side.

---

* The arc 66° 3′ is subtracted from 180°, because the sum of the sides a and b is greater than 180°. (See note, p. 69.)

Under [1], log. cosec. } of half the sum of the given angles
Under [2], log. sec. } (rejecting 10 in each index).
Under [1], log. sin. } of half the difference of given angles.
Under [2], log. cos. }

Add together logarithms under [1] and [2], and (rejecting 10 in each index) the results will be log. tangents of half the difference and half the sum of required sides respectively (*a*). The sum of the two arcs will be the side opposite the greater of the given angles, and their difference will be the side opposite to the less of the given angles.*

(*a*) *Note.* If half the sum of the given angles exceed 90°, the arc taken out under [2] must be subtracted from 180°, to give half the sum of the required sides.

EXAMPLE.

222. In the spherical triangle ABC, given A=113° 33′ 30″, B=51° 30′ 20″, and c=60° 18′ 4″: required the other two sides.

A=113° 33′ 30″
B= 51 30 20
A+B=165 3 50
A−B= 62 3 10

c=60° 18′ 4″
∴ ½ c=30 9 2
½ (A+B)=82 31 55
½ (A−B)=31 1 35

[1].

tan. ½ c	9·764070
cosec. ½ (A+B)	0·003700
sin. ½ (A−B)	9·712173
tan. ½ $(a-b)$	9·479943
½ $(a-b)$	16° 48′

[2].

tan. ½ c	9·764070
sec. ½ (A+B)	0·886220
cos. ½ (A−B)	9·932952
tan. ½ $(a+b)$	10·583242
½ $(a+b)$	75° 22′
½ $(a-b)$	16 48
a=	92 10
b=	58 34

---

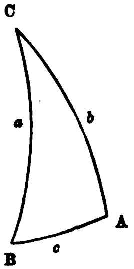
* By formula U (Part II.)

$$\tan. \tfrac{1}{2}(a+b)=\frac{\cos. \tfrac{1}{2}(A-B)}{\cos. \tfrac{1}{2}(A+B)} . \tan. \tfrac{1}{2} c$$

$$\tan. \tfrac{1}{2}(a-b)=\frac{\sin. \tfrac{1}{2}(A-B)}{\sin. \tfrac{1}{2}(A+B)} . \tan. \tfrac{1}{2} c$$

Rule XIII.

RIGHT-ANGLED SPHERICAL TRIANGLES.



In a right-angled spherical triangle ABC, A being the right angle, the two sides adjacent to the right angle *b* and *c*, and the *complements** of the three other parts, namely, co. *a*, co. B, and co. C, are called the *five circular parts*. By the following rules, called, from the name of their author, *Napier's Rules*, two of these parts being given, any one of the other three may be found.

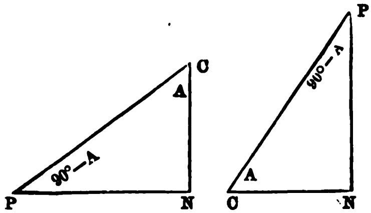
In applying the rules, we must select one of the three parts concerned, such that the other two may either be *both adjacent* to it, or *both separate* from it. The part so selected is called the *middle part*. Rule A will apply to the former case, Rule B to the latter.

Rule A.

The *sine* of the middle part is equal to the product of the *tangents* of the two parts *adjacent* to it.

---

* The *complement* of an angle is what it wants of 90°; thus, in the right-angled triangle PNC (1st fig.), the angle P, or 90°−A, is the complement of the angle A. It may easily be proved that the sin., tan., sec., cos., cot., and cosec. of an angle are the cos., cot., cosec., sin., tan., and sec. of its complement respectively. Thus, let the two triangles CPN be equal to each other in every respect, then P is the complement of A, or co. A.



Now [Art. (5), p. 3] sin. A$=\frac{PN}{CP}=$cos. (90°−A)=cos. co. A

tan. A$=\frac{PN}{CN}=$cot. (90°−A)=cot. co. A, &c.

Hence, in the above rules, sin. A is substituted for cos. co. A; cos. A for sin. co. A; cot. A for tan. co. A, &c. (See Part II.)

Rule B.

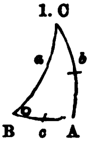
The *sine* of the middle part is equal to the product of the *cosines* of the two parts *opposite to*, or *separated from* it.*

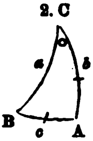
Having written down the equation according to the case, make a dash under the part required, and determine its magnitude by applying the proper signs (+ or —) to each term (Rule XVI. p. 30).

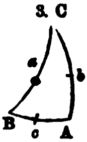
EXAMPLES.

In a right-angled spherical triangle ABC, given $b=74° 19' 30''$, $c=38° 56'$, and $A=90°$: required the other three parts.

Make a triangle for each of the three required parts, and mark the two given parts and the unknown part as directed (p. 48), and then consider which of the three is the middle part, and whether the other two marked parts are adjacent, or opposite parts.







Thus in fig. 1, to find B, the right angle A not being considered as a part, c will be the middle part, and complement B and b are the *adjacent* parts.

$$\therefore \text{ By Rule A, } \sin. c=\tan. \text{co. } B . \tan. b$$

$$\text{or } \sin. c=\underline{\cot. B} . \tan. b$$

(since tan. co. B=cot. B by note, p. 72): and determining the sign of B by Rule XVI. p. 30, we have

$$\overset{+}{\sin.}\ c=\underline{\overset{+}{\cot.}\ B} . \overset{+}{\tan.}\ b$$

In logarithms,

$$\log. \sin. c-10=\log. \cot. B-10+\log. \tan. b-10$$

$$\text{whence } \log. \cot. B=\log. \sin. c+10-\log. \tan. b$$

---

* For verification of these rules, see p. 72, Part II.

Calculation.

$b=74° 19' 30''$
$c=38\ \ 56$

log. sin. $c+10$	19·798247
„ tan. b	10·551887
„ cot. B	9·246360

$\therefore B=80°$

In fig. 2, to find C. Marking the figure in the usual manner, we see that b is the middle part, and c and C the adjacent parts.

∴ By Rule A,

$\sin. b=\tan. c\ .\ \tan. \text{co.} C$

$+\qquad+\qquad+$

or $\sin. b=\tan. c\ .\ \underline{\cot. C}$

Calculation.

log. sin. $b+10$..	19·983540
„ tan. c	9·907336
„ cot. C	10·076204

$\therefore C=40°$

In fig. 3, to find a. In this case a is the middle part, and b and c are the parts separated from it.

∴ By Rule B,

$\sin. \text{co.} a=\cos. b\ .\ \cos. c$

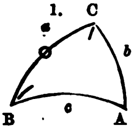
$+\qquad+\qquad+$

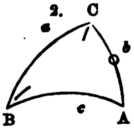
or $\underline{\cos. a}=\cos. b\ .\ \cos. c$

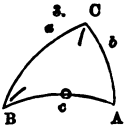
Calculation.

log. cos. b......	9·431654
„ cos. c......	9·890911
log. cos. a (rejecting 10)	9·322565

$\therefore a=77° 52''$

1.


2.


3.


Given A= 90°, B =78° 10′ and C= 131° 32′ 45″: to find the other parts.

To find a (fig. 1).

$-\qquad+\qquad-$

$\underline{\cos. a}=\cot. B\ .\ \cot. C$

log. cot. B .	9·321222
„ cot. C .	9·947508
„ cos. a .	9·268730

$79° 18' 0''$
180
$\therefore a=100\ \ 42\ \ 0$

To find b (fig. 2).

$+\qquad+\qquad+$

$\cos. B=\underline{\cos. b}\ .\ \sin. C$

cos. B+10 .	19·311893
sin. C	9·874148
cos. b	9·437745

$\therefore b=74° 5' 45''$

To find c (fig. 3).

$-\qquad+\qquad-$

$\cos. C=\sin. B\ .\ \underline{\cos. c}$

cos. C+10 .	19·821657
sin. B	9·990671
cos. c	9·830986

$47° 20' 30''$
180
$\therefore c=132\ \ 39\ \ 30$

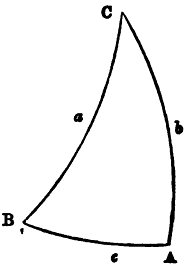
In a right-angled spherical triangle ABC, find the other parts, having given:

223.	B= 72° 19′ 0″	*Ans.*	a= 54° 28′ 0″
	b= 50 50 0		c= 23 2 15
	A= 90 0 0		C= 28 45 0
224.	b= 60 10 0	”	B= 60 32 45
	c=100 0 0		C= 98 41 45
	A= 90 0 0		a= 94 57 15
225.	B=100 0 0	”	a= 90 30 0
	C= 87 10 0		b=100 0 45
	A= 90 0 0		c= 87 7 15
226.	c= 46 18 23	”	b= 26 23 15
	B= 34 27 30		a= 51 46 15
	A= 90 0 0		C= 66 59 30
227.	c=118 21 4	”	b=116 18 0
	A= 23 40 12		C=100 59 30
	B= 90 0 0		a= 21 5 45
228.	a=100 42 0	”	C=131 32 45
	B= 78 10 0		c=132 39 30
	A= 90 0 0		b= 74 5 45
229.	c= 53 14 20	”	b= 91 4 15
	A= 91 25 58		C= 53 15 0
	B= 90 0 0		a= 91 47 15

RULE XIV.

QUADRANTAL SPHERICAL TRIANGLES.

A spherical triangle having one *side* 90° is called a *quadrantal* triangle.



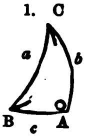
The five circular parts in a quadrantal triangle are the two angles adjacent to the quadrant, and the complements of the other three; thus, if ABC be a quadrantal triangle, the side a being 90°, the five circular parts are the angles adjacent to the quadrant B and C, and the complements of the angle A and of the sides b and c.

Any two of these parts being given, a third may be found by the preceding rules for right-angled triangles (p. 72).

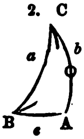
The magnitude (whether greater or less than 90°) of the part required is determined by Rule XVI. p. 30, observing, that *when two angles or two sides come together on the same side of the equation*, the sign — must be placed before them, and the three signs thus placed on one side of the equation must be made to produce the same result (positive or negative) as the sign of the other side.*

EXAMPLES.

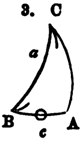
1.



2.



3.



In the quadrantal triangle ABC, given B = 80° 10′, C = 48° 50′, and a=90: find the other parts.

To find A (fig. 1).	To find b (fig. 2).	To find c (fig. 3).
$\underline{\overset{-}{\cos. A}} = -\overset{+}{\cos. B} . \overset{+}{\cos. C}$	$\overset{+}{\sin. C} = \overset{+}{\tan. B} . \underline{\overset{+}{\cot. b}}$	$\overset{+}{\sin. B} = \overset{+}{\tan. C} . \underline{\overset{+}{\cot. c}}$
log. cos. B . 9·232444	sin. C+10 . 19·876678	sin. B+10 . 19·993572
„ cos. C . 9·818392	tan. B . . . . 10·761128	tan. C . . . . 10·058287
„ cos. A . 9·050836	cot. b . . . . . 9·115550	cot. c . . . . . 9·935285
83° 32′ 45″	b=82° 34′	c=49° 15′ 51″
180		
A=96 27 15		

In a quadrantal spherical triangle ABC, find the other parts, having given:

230.	a= 90°	0′	0″	*Ans.*	B= 74°	36′	30″
	A=100	0	0		b= 78	14	30
	c= 50	10	0		C= 49	8	0
231.	B= 45	0	0	„	A=107	10	15
	c= 72	0	0		b= 47	44	30
	a= 90	0	0		C= 65	19	15

---

* For verification of rules for quadrantal triangles, see p. 74, Part II.

232.	$b=$	90°	0′	0″	*Ans.*	B=	88°	36′	45″
	$a=$	100	0	0		A=	100	5	45
	$c=$	82	10	30		C=	82	3	15
233.	$a=$	90	0	0	„	A=	96	17	45
	B=	80	10	0		$b=$	82	26	0
	C=	50	2	0		$c=$	50	27	0
234.	A=	72	49	45	„	C=	114	40	45
	$b=$	47	44	30		B=	44	59	45
	$a=$	90	0	0		$c=$	108	0	0
235.	$c=$	49	23	45	„	A=	101	42	15
	$b=$	76	41	0		B=	72	20	30
	$a=$	90	0	0		C=	48	1	50
236.	$a=$	60	10	15	„	A=	59	41	45
	$b=$	80	20	30		B=	78	51	0
	$c=$	90	0	0		C=	95	36	0

EXAMINATION PAPERS IN PRECEDING RULES.

I.

237. Required the product of $18 \times 48 \times 6{\cdot}2 \times 4$ *Ans.* 21427
238. Divide 236 by 16·1 „ 14·66
239. Required the 3d power of 12·5 „ 1953·125
240. „ 4th power of ·076543 „ ·000034326
241. „ 6th root of 11078 „ 4·72146
242. „ 7th root of ·098674 „ ·7183146
243. „ value of $(19)^{\frac{2}{5}}$ „ 3·247
244. „ value of x in the following proportion:
$24 : 17{\cdot}5 :: 79 : x$ *Ans.* $x=57{\cdot}604$

245. Reduce the following expression to logarithms: $x=\frac{a^2b}{cd}$ *Ans.* log. $x=2$ log. $a+$log. $b-$log. $c-$log. d

246. Find the value of x in the following equation: $2^x=769$. *Ans.* $x=9{\cdot}5868$

247. The first term of a geometrical series is 2, its common ratio 3, and number of terms 8; find the sum. *Ans.* S=6560

248. If 400*l.* be placed out at compound interest for nine years, at 4*l.* per cent. per annum, required its amount. *Ans.* 569*l.* 6*s.* 8*d.*

II.

249. In the plane triangle ABC, given $a=10$, $b=15$, and $c=20$; required the angle A. *Ans.* A=28° 57′ 15″

250. In the plane triangle ABC, given $a=25{\cdot}125$, $b=12\frac{1}{2}$, and A=68° 48′; to find B. *Ans.* B=27° 38′ 15″

251. Given $a=399$, $b=230$, and C=55° 2′ 15″; required the angle A. *Ans.* A=89° 45′ 37″

252. Given $a=40$, $b=60$, and C=100°; required c. *Ans.* $c=77{\cdot}68$

253. In the right-angled triangle ABC, given B=90°, $a=210$, and A=50° 10′; required the other parts. *Ans.* C=39° 50′, $c=175{\cdot}1$, $b=273{\cdot}5$

254. Given $a=399$, $b=230$, and $C=124° 57' 45''$; required the area. *Ans.* 37604

255. Given $a=399$, $b=230$, and $c=327$; required the area. *Ans.* 37604·7

256. Give definitions of the sine, cosine, tangent, cotangent, secant, cosecant, and versine of an angle.

III.

257. In the spherical triangle ABC, given $a=120° 54'$, $b=105° 6'$, and $c=108° 41' 30''$; required the angles A, B, and C. *Ans.* $A=130° 50'$, $B=121° 35'$, and $C=123° 18'$

258. In the spherical triangle ABC, given $a=64° 21' 15''$, $b=80° 38' 45''$, and $c=104° 28' 30''$; required the angles A, B, and C.

Ans. $A=60° 17' 45''$, $B=71° 55' 45''$, and $C=111° 6' 15''$

259. Given $a=87° 10' 15''$, $c=100° 10' 15''$, and $B=61° 31' 15''$; required b. *Ans.* $b=62° 36' 45''$

260. Given $a=81° 10'$, $A=81° 24'$, and $b=62° 36' 45''$; required B. *Ans.* $B=62° 41'$

261. Given $a=49° 10'$, $b=58° 25'$, and $C=71° 18' 30''$; required the angles A and B. *Ans.* $A=59° 2'$, $B=74° 54'$

262. Given $a=87° 10' 15''$, $c=100° 10' 15''$, and $B=61° 31' 15''$; required the angles A and C.

Ans. $A=81° 24'$, $C=102° 59'$

263. Given $a=59° 2'$, $B=74° 54'$, and $c=56° 42'$; required the sides a and b. *Ans.* $a=49° 10'$, $b=58° 25'$

264. Given $A=115° 38' 45''$, $C=75° 31' 30''$, and $b=108° 4' 15''$; required the sides a and c.

Ans. $a=119° 42' 15''$, $c=68° 53' 45''$

IV.

265. In the right-angled spherical triangle ABC, given $B=60° 32' 15''$, $a=94° 57' 20''$, and $A=90°$; required the other parts. *Ans.* $C=98° 41' 45''$, $c=100°$, $b=60° 10'$

266. Given $a=77° 52' 10''$, $b=74° 19' 30''$, and $A=90°$; to find the other parts.

Ans. $c=38° 56' 24''$, $B=80°$, $C=40°$

267. In the quadrantal triangle ABC, given $a=90^\circ$, $B=74^\circ\ 36'\ 30''$, and $c=50^\circ\ 10'$; required the other parts.
Ans. $A=100^\circ$, $b=78^\circ\ 14'\ 25''$, $C=49^\circ\ 8'\ 15''$

268. Given $a=90^\circ$, $A=96^\circ\ 17'\ 52''$, $b=82^\circ\ 26'$; required the other parts. *Ans.* $B=80^\circ\ 10'$, $C=50^\circ\ 2'$, $c=50^\circ\ 27'$

V.

269. Required the product of $2{\cdot}4 \times {\cdot}0035 \times 1{\cdot}08 \times {\cdot}1$.
Ans. ·0009072

270. Divide 9·5 by ·36. *Ans.* 26·389

Find the value of $\dfrac{8{\cdot}4 \times {\cdot}0769 \times {\cdot}00683}{59{\cdot}8 \times {\cdot}0000146 \times {\cdot}0039}$
Ans. 1295·71

271. Find the 3d power of ·2321 and the 72d power of ·96797. *Ans.* ·0125 and ·096

272. Required the 200th root of ·0063241. *Ans.* ·975

273. Required the value of $\sqrt[{\cdot}07]{{\cdot}6958825}$. *Ans.* ·00563

274. Required the value of the following expression:

$$\sqrt{\frac{\sqrt[{\cdot}72]{{\cdot}096}+({\cdot}096)^{{\cdot}72}}{2{\cdot}5}}$$

Ans. ·29904

275. In the plane triangle ABC, given $a=\frac{1}{8}$, $b={\cdot}2705$, and $c={\cdot}3375$; required the angle C. *Ans.* $C=111^\circ\ 47'\ 45''$

276. In the plane triangle ABC, given $a=116$, $b=172{\cdot}5$, and $A=37^\circ\ 20'$; to find B. *Ans.* $B=64^\circ\ 24'$, or $115^\circ\ 36'$

277. In the plane triangle ABC, given $a={\cdot}512$, $b={\cdot}627$, and $C=42^\circ\ 53'\ 38''$; required the angles A and B.
Ans. $A=54^\circ\ 8'\ 11''$, $B=82^\circ\ 58'\ 11''$

278. Given $b={\cdot}2$, $c={\cdot}25$, and $A=22^\circ\ 20'$; required the third side a. *Ans.* $a={\cdot}1$

279. In the right-angled plane triangle ABC, given $a=177\frac{3}{4}$, $c=117{\cdot}7$, and $B=90^\circ$; required the side b.
Ans. $b=213{\cdot}21$

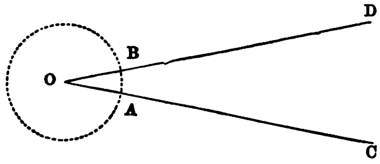
280. In the trapezium ABCD (fig. p. 61), given the side AB=90 yards, BC=100 yards, CD=110 yards, DA=120 yards, and the angle DAB=116°; required the area.
Ans. Area=9768·7 yards

CHAPTER IV.

APPLICATION OF THE RULES AND FORMULÆ OF PLANE AND SPHERICAL TRIGONOMETRY TO THE SOLUTION OF PROBLEMS IN ASTRONOMY, SURVEYING, AND NAVIGATION.

(37). When a line joining any two points in space is accessible throughout its whole extent, it may in general be measured by the successive application of some line of a known length; but when it is inaccessible, or cannot be directly measured, we may obtain its length by considering it the *side of a triangle*, if we already know, or can find by observation, a sufficient number of parts of that triangle to enable us to apply one or more of the preceding rules. It is thus that the mensuration of inaccessible lines of any length is found by means of that of accessible lines and angles.

The angle DOC, contained by lines drawn from a point O to two remote objects C and D, may be measured by placing a circle in the plane passing through the two objects, and having its centre at the angular point O. The straight edge of a ruler being then placed on the circle so as to pass through its centre, and, by means of sights placed over it, directed first to one object and then to the other, the arc AB of the circumference between the two positions of the ruler can be found; this is the measure of the angle O.



(38). The principal instruments for measuring angles are the theodolite and sextant. A theodolite is the most convenient instrument for measuring horizontal and vertical angles: it is composed of two circles having their planes perpendicular to each other. When the instrument is used, one of the circles is placed in a horizontal plane, by means of levels; on this circle horizontal angles are measured; on the other are measured vertical angles, whether of elevation or depression (that is, whether the object is above or below the horizontal line). A Sextant is employed to measure angles contained in any plane whatever. It is more suited for observing angular distances of heavenly bodies than the theodolite; but the latter is better adapted for Surveying than the former, since it determines the horizontal angles at once; but those observed with the sextant must, when out of the plane of the horizon, be reduced to that plane by calculation, to suit them to the purposes of the survey.

We will not stop to give particular descriptions of the theodolite and sextant: their construction will be best learned by a careful study of the instruments themselves; we shall therefore suppose the manner of adjusting and applying them to practice is known,* and proceed to give a collection of problems in which these instruments have supplied for the most part the necessary data.

SECTION I.

Problems in Surveying, &c.

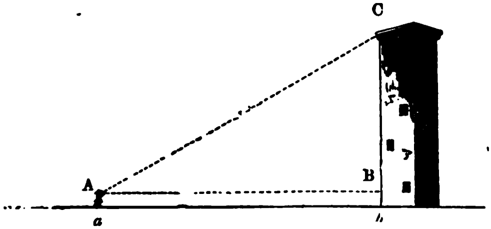
The rules of Plane Trigonometry just given will enable us to solve many useful and interesting problems in Surveying and Navigation. A large collection of these will be found at the end of the book; they have been selected to serve as an introduction to Navigation and Nautical Astronomy. We will give, however, in this place a few that most frequently occur in practice, with their solutions at length; and as it is often very necessary that a naval student should know how to solve trigonometrical problems, not only by logarithms, but also *instrumentally*, that is, by constructing the figures with the aid of mathematical instruments, we will make use of both methods when we come to solve some of the problems in Navigation.

---

* The adjustments and method of using the sextant, the instrument chiefly used by the seaman for measuring angles, are fully described in Navigation, Part I.

1. Wishing to find the height of a tower, I observed the angle of elevation of its top above the level of my eye to be 32° 14′. I then measured the distance *ab* from the place of observation to the base of the tower, and found it to be 142 feet. Required the height of the tower.

Let CB represent the tower, and A the place of the observer: draw the horizontal line AB at a height above the ground equal to the height of the eye, and join AC. Then in the right-angled triangle ABC are given AB=142 feet, the angle CAB=32° 14′, and the angle B=90°; to find CB, the height of the tower above the horizontal line AB.



By Rule, p. 57, $\frac{CB}{AB}=\text{tan. } A$

$\therefore CB=AB \text{ tan. } A.$

$AB=142$ $\qquad \therefore \text{log. } CB=\text{log. } AB+\text{log. tan. } A-10$

$A=32° 14′$ $\qquad$ „ AB .... 2·152288

„ tan. A...... 9·799717

„ CB .... 1·952005

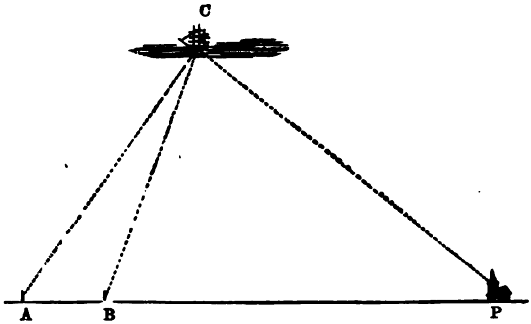
$\therefore CB=89{\cdot}5$ feet.

To the value of CB must be added the height of the eye *Aa*: the result will be the height of the tower required.

2. Being on Southsea Common, and wishing to find my

distance from a ship at anchor at Spithead, I observed with a sextant the angle between the ship and the steeple of St. Thomas's Church to be 72° 42′. I then walked 500 yards in a direct line towards the church, and again took the angle between the ship and the steeple, and found it to be 82° 45′. Required my distance from the ship at each observation.

Let C be the ship at anchor, A my first station, B the second, and P the church. Then the angle CAP=72° 42′,



CBP=82° 45′, and the line AB=500 yards, are given to find AC and BC, my distances at each observation.

In the triangle ABC, the angle ABC=180°—CBP=97° 15′
. ·. angle C=180° – (72° 42′+97° 15′)=10° 3′.

(1.) To find BC.

By Rule, p. 47, $\underline{\text{BC}}$: AB : : sin. A : sin. C.

AB=500	log. AB	2·698970
A=72° 42′	„ sin. A	9·979895
C=10 3		12·678865
	„ sin. C	9·241814
	„ BC	3·437051

. ·. distance BC=2735 yards.

(2.) To find AC.

$\underline{\text{AC}}$: AB : : sin. ABC : sin. C

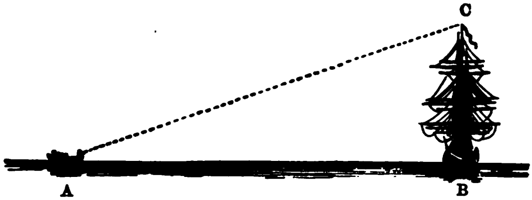
AB=500
ABC=97° 15′
C=10 3

log. AB	2·698970
„ sin. ABC....	9·996514
	12·695484
„ sin. C	9·241814
„ AC...........	3·453670

. ·. distance AC=2841 yards.

3. Being ordered to place a target at 500 yards from the ship, and knowing that the height of the truck above the water-line was 213 feet, it is required to find what angle this height will subtend on my sextant when I am at the required distance (before allowing for index correction of instrument).

Let BC represent the ship's mast, A the required place of the target: then the angle BAC is the angle which must be read off on the sextant (supposing it to have no ind. cor.).



In the right-angled triangle ABC are given the side BC= 213 feet, AB=500 yards, or 1500 feet, and B=90°; to calculate the angle A.

By Rule, p. 57, $\tan. A = \frac{BC}{AB}$

BC= 213
AB=1500

. ·. log. tan A−10=log. BC−log. AB
or log. tan. A=10+log. BC−log. AB

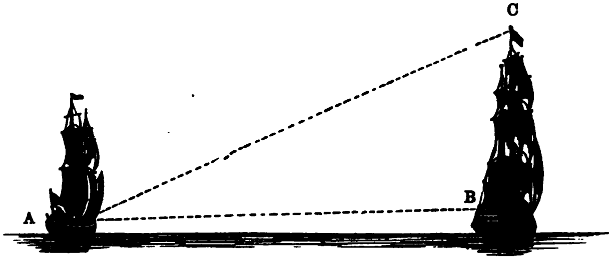
log. BC+10.........	12·328380
„ AB..............	3·176091
„ tan. A	9·152289

. ·. A=8° 5′

4. Sailing in company with another ship, and being ordered to keep at the distance of 500 yards from her, and knowing that the height of her mast above the hammock nettings was

198 feet, it is required to find what angle on my sextant will indicate the proper distance.

Let A and B be the two ships, A the place of the observer, and CAB the angle subtended by the mast CB at A. Then in the right-angled triangle CAB are given CB=198 feet, AB=1500 feet, and B=90°; to find the angle CAB.



By Rule, p. 57, $\tan. A = \frac{CB}{AB}$

CB= 198 $\quad$.·. log. tan. A−10=log. CB−log. AB

AB=1500 $\quad$ log. tan. A=10+log. CB−log. AB

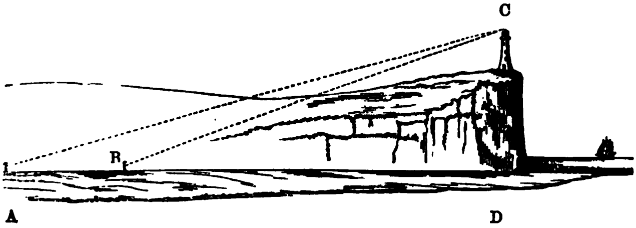
log. CB+10......... 12·296665

„ AB............... 3·176091

„ tan. A .... 9·120574

.·. A=7° 31′ 15″

5. Wishing to determine the height of a lighthouse C on the summit of a cliff on the sea-shore, I observed the angle of elevation CAD of its top above the level sand to be 26° 40′;



then, measuring on the sand in a direct line towards it a line AB=200 yards, I again observed the angle of elevation of its top, and found it to be 33° 30′. Required the height CD of the lighthouse above the shore.

(1.) In the triangle CAB are given AB=200, angle CAB =26° 40′, angle ABC=180°−33° 30′=146° 30′; and therefore the remaining angle ACB=6° 50′; to find the side CB.

By Rule, p. 47, $\underline{\text{CB}}$: AB :: sin. A : sin. ACB

AB=200 log. AB .... 2·301030

A=26° 40′ ,, sin. A .... 9·652052

ACB= 6 50 11·953082

,, sin. ACB ..... 9·075480

,, CB .... 2·877602

∴ CB=754·4 yards.

(2.) In the right-angled triangle CBD are given log. CB =2·877602, angle CBD=33° 30′, and D=90°; to find CD, the height of the lighthouse.

By Rule, p. 57, $\frac{\text{CD}}{\text{CB}}$=sin. CBD ∴ CD=CB sin. CBD

∴ log. CD=log. CB+log. sin. CBD—10

log. CB............... 2·877602

,, sin. CBD ...... 9·741889

12·619491

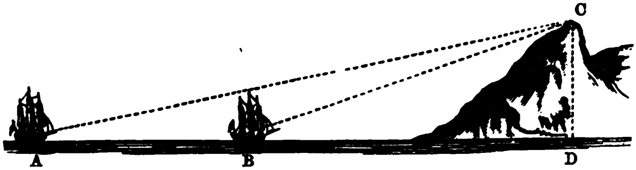
10

,, CD .... 2·619491

∴ CD=416·3 yards.

6. Standing in for the land, I observed the summit of a lofty mountain near the shore. I took the angle of elevation of the peak, and found it to be 12° 25′; after having run 3½ miles directly towards it, I again took the angle of elevation, which was 30° 13′. Required the height of the mountain, and its distance from the second station.

Let CD represent the mountain, A the ship's place at the first observation, B the place at the second observation. Then, in the triangles CAB and CBD we have given the side



AB=3·5 miles, the angle CAB=12° 25′, and the angle CBD=30° 13′; to find the height of the mountain CD, and the distance DB from the station B.

(1.) To find BC.

In the triangle ABC, BC : AB : : sin. BAC : sin. ACB.

AB=3·5	log. AB	0·544068
BAC=12° 25′	,, sin. BAC	9·332478
ACB=CBD−BAC		9·876546
CBD=30° 13′	,, sin. ACB	9·485289
BAC=12 25	,, BC	0·391257
∴ ACB=17 48	∴ BC=2·46 miles.	

(2.) To find CD.

In the right-angled triangle CBD, $\frac{CD}{BC}$=sin. CBD

∴ CD=BC . sin. CBD

BC=2·46	log. BC	0·391257
CBD=30° 13′	,, sin. CBD ...	9·701802
	,, CD	0·093059
	∴ CD=1·24 miles.	

(3.) To find DB.

In the right-angled triangle CBD, $\frac{BD}{BC}$= cos. CBD

∴ BD=BC . cos. CBD

BC=2·46

CBD=30° 13′

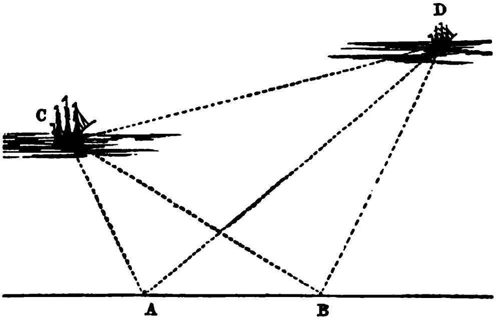
log. BC .... 0·391257

„ cos. CBD ... 9·936578

„ BD .... 0·327835

. ·. BD=2·13 miles.

7. Wishing to know the distance between two ships at anchor at C and D, I measured on the shore a base line AB=735 feet, and with a sextant observed the following angles.



At A, one end of the base, CAD=63° 30′, and DAB=35° 10′; at B, the other end of the base, DBC=80° 16′, and CBA=28° 20′. Required CD, the distance between the two ships.

(1.) In triangle ACB, find AC.

(2.) In triangle ABD, find AD.

(3.) In triangle ACD, find CD.

(1.) To find AC.

In triangle ACB are given AB=735, CBA=28° 20′, CAB=63° 30′+35° 10′=98° 40′, and therefore the remaining angle ACB=53°.

CBA= 28° 20′

CAB= 98 40

127 0

180

ACB= 53

AB=735

AC : AB : : sin. CBA : sin. ACB

log. AB .... 2·866287

„ sin. CBA .... 9·676328

12·542615

„ sin. ACB ..... 9·902349

„ AC .... 2·640266

. ·. AC=436·8

(2.) To find AD.

In the triangle ABD are given AB=735, DAB=35° 10′, ABD=28° 20′+80° 16′=108° 36′, and therefore the remaining angle ADB=36° 14′.

DAB=	35° 10′
ABD=	108 36
	143 46
	180
ADB=	36 14
AB=	735

$\underline{\text{AD}}$: AB : : sin. ABD : sin. ADB

log. AB	2·866287
„ sin. ABD	9·976702
	12·842989
„ sin. ADB	9·771643
„ AD	3·071346

∴ AD=1178·5

(3.) To find CD.

In the triangle ACD are given AC=436·8, AD=1178·5, and the included angle CAD=63° 30′.

By Rule IV. second method, p. 55.

AD.........	1178·5
AC.........	436·8
AD+AC..	1615·3
AD−AC..	741·7

CAD............	63° 30′
∴ ½ CAD...	31 45

[1.]

log. (AD+AC)...	3·208253
„ (AD−AC)...	2·870226
	0·338027
„ tan. ½ CAD..	9·791563
„ tan. arc.	10·129590

[2.]

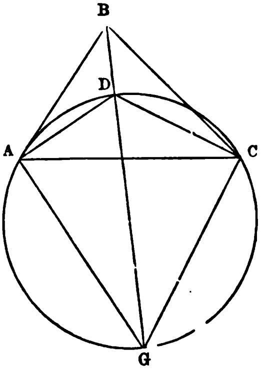
log. (AD+AC)...	3·208253
„ sin. ½ CAD...	9·721162
	12·929415
„ sin. arc	9·904757
„ CD............	3·024658

∴ CD=1058·5

As the two following problems are of great use in Marine Surveying, we will solve them by logarithms, and also by a geometrical construction. In problem 98 of the volume of Astronomical Problems, analytical solutions of the same problems are also given.

8. Wishing to determine the position of a sunken rock at

the entrance of a bay, and the water being smooth, I anchored a boat upon it, and measured with a sextant the angles which three objects, A, B, and C, on the shore subtended at the boat. They were as follows: the angle between A and the object B to the right was 26° 27′, and the angle between B and the object C to the right was 34° 12′. On my chart of the bay I carefully measured with compasses the distances between the three objects, and found AB=5 miles, BC=6 miles, and AC=7 miles. Required the distance of the rock from A, B, and C.



(1.) *By Construction.*

By means of a scale of equal parts make the triangle ABC, having the side AB=5, BC=6, and AC=7. At the point C, on the side of AC farthest from the boat, make the angle ACD=26° 27′, the angle observed between the other two objects A and B; and at the point A, on the farthest side also from the boat, make the angle CAD=34° 12′, the angle between the other two objects B and C: produce the sides AD and CD till they meet in D. Then describe a circle to pass through the three points A, D, and C; and the position of the rock will be somewhere in the circumference of that circle. To find that position, join BD, and produce it to the circumference in G; then G will be the station sought, or the position of the rock.

For, the angles in the same segment of a circle being equal (*Euclid*, b. iii.), therefore AGB=ACD=26° 27′, and CGB=CAD=34° 12′; and these were the angles observed at the boat. Hence G must be the position of the boat; and GA, GB, and GC measure respectively the distance of the rock from each of the objects A, B, and C.

(2.) *By Trigonometry.*

Assume any point G to be the position of the boat, and let A, B, and C be the objects. Describe roughly a circle passing through the three points A, G, C. Join GA, GB, and GC. Then GA, GB, and GC are the distances required. Draw AD, CD to the point of intersection D. Then, by Geometry, since the angles in the same segment of a circle are equal, ∴ CAD=CGB=34° 12′, and ACD=AGB=26° 27′.

[1.] Find AD, having given in the triangle ADC the side AC=7, the angle ACD=26° 27′, and ADC=180°−(34° 12′ +26° 27′)=119° 21′.

[2.] Find angle BAC, having given the three sides of the triangle ABC.

[3.] Find angle ABD, having given AB, AD, and angle BAD (=BAC−CAD).

[4.] Find GA, GB, and angle BAG, having given in the triangle ABG the side AB and the angles AGB, and ABG.

[5.] Find GC, having given in the triangle AGC the side AC, the angle AGC, and the angle CAG(=BAG−BAC).

Calculation.

[1.] To find AD.

AC : AD :: sin. ADC : sin. DCA

AC=7	0·845098
ADC=119° 21′	9·648766
DCA= 26 27	10·493864
	9·940338
	0·553526

∴ AD=3·577

[2.] To find angle BAC.

	7	9·154902
AB=5	5	9·301030
BC=6	2	0·602060
AC=7	6	0·301030
	8	9·359022
	4	BAC=57° 7′ 15″
	4	CAD=34 12 0
	2	∴BAD=22 55 15

[3.] To find angle ABD. Rule III., p. 52.

AB+AD:AB−AD::tan.½(ADB+ABD):tan.½(ADB−ABD)

AB=5	0·153205
AD=3·577	10·692995
8·577	10·846200
1·423	0·933335
BAD= 22° 55′ 15″	9·912865
180	39° 17′ 15″
157 4 45	78 32 22
78 32 22	ADB=117 49 37
	ABD= 39 15 7

[4.] To find GA, GB, and the angle BAG.

AB : GA : : sin. AGB : sin. ABG

AB : GB : : sin. AGB : sin. BAG

AB=5		
AGB= 26° 27′		
ABG= 39 15	0·698970	0·698970
65 42	9·801201	9·959711
180	10·500171	10·658681
BAG=114 18	9·648766	9·648766
	0·851405	1·009915
	GA=7·103	GB=10·23

[5.] To find GC.

AC : GC : : sin. AGC : sin. CAG

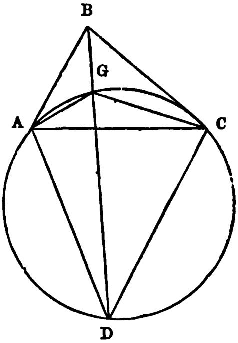
AC=7	0·845098
AGC= 60° 39′	9·924491
BAG=114 18	10·769589
BAC= 57 7	9·940338
∴ CAG= 57 11	0·829251
	∴ GC=6·75

9. Sailing in a deep and unknown bay, I suddenly found the soundings decrease; and suspecting I was on a coral reef, I hauled off, having anchored a boat on it, from which the following angles were taken between three remarkable objects, A, B, and C, that appeared on the distant shores, namely: between A and B, a high-pointed rock to the right of A, the angle was 116° 40′; between B and C, a bluff summit to the

right of B, the angle was 112° 30′. I obtained afterwards the distances between the three points A, B, and C, by measuring on the shore convenient base lines, and taking angles, as pointed out in Example, p. 89. The distance from A to B was 5·75 miles, from B to C 7·5 miles, and from A to C 8·25 miles. It is required to find the position of the reef.

(1.) *By Construction.*

Construct the triangle ABC from the scale of equal parts, by taking AB=5·75, BC=7·5, and AC=8·25. At the point C, in the straight line AC, make ACD=63° 20′, the supplement of the angle subtended by the other two points A and B. Again, at the point A, in the straight line AC, make CAD=67° 30′, the supplement of the angle subtended by the other two points B and C: produce the lines AD and CD to meet in the point D. About the triangle ADC describe a circle; then the place of the reef will be somewhere in the circumference of this circle. To find it, join BD; and the point of intersection G is the position of the reef required.



For since the angles in the same segment of a circle are equal (*Euclid*, b. iii.), therefore AGD=ACD=63° 20′; therefore the angle AGB=116° 40′. Again, CGD=CAD= 67° 30′; therefore the angle BGC=112° 30′. And these were the angles observed at the boat; therefore G must be the place of the boat, or position of reef.

(2.) *By Trigonometry.*

Assume any point G as the position of the reef, and let A, B, and C be the objects on shore. Describe a circle passing through the three points A, G, and C. Join BG, and produce it to meet the circle in D. Join GA and GC. Then GA, GB, and GC are the distances required. Join AD and CD. Then, by Geometry, since the angles in the same segment are equal, ∴ angle DAC=DGC=180°−BGC=180°−112° 30′=67° 30′, and angle ACD=AGD=180°−AGB=180°−116° 40′=63° 20′.

[1.] Find AD, having given in the triangle ADC, AC=8·25, angle ACD=63° 20′, and angle ADC=180°−(67° 30′+63° 20′)=49° 10′.

[2.] Find the angle BAC, having given the three sides of the triangle ABC.

[3.] Find the angle ABD, having given AB, AD, and the angle BAD(=BAC+CAD).

[4.] Find GA, GB, and the angle BAG, having given in the triangle ABG the side AB and the angles AGB and ABG.

[5.] Find GC, having given in the triangle AGC, the side AC, the angle AGC, and the angle CAG(=BAC−BAG).

Calculation.

[1]. To find AD.

AC : $\underline{\text{AD}}$: : sin. ADC : sin. DCA

AC=8·25	0·916454
ADC=49° 10′	9·951159
DCA=63 20	10·867613
	9·878875
	0·988738
	∴ AD=9·744

[2.] To find the angle BAC (p. 44).

AB=5·75	8·25	9·083546
AC=8·25	5·75	9·240332
BC=7·5	2·50	0·698970
	7·5	0·397940
	10·0	9·420788
	5·0	BAC= 61° 46′ 15″
	5·0	CAD= 67 30 0
	2·5	BAD=129 16 15

[3]. To find the angle ABD.

$$AD+AB:AD-AB::\tan.\tfrac{1}{2}(ABD+ADB):\underline{\tan.\tfrac{1}{2}(ABD-ADB)}$$

AD= 9·744	0·601408
AB= 5·750	9·675890
15·494	10·277298
3·994	1·190164
180° 0′ 0″	9·087134
BAD=129 16 15	6° 58′
50 43 45	25 22
25 21 52	∴ ABD=32 20

[4.] To find GA, GB, and the angle BAG.

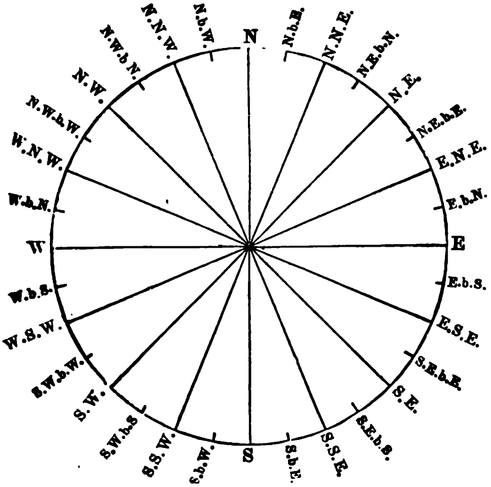
AB=5·75	AB : $\underline{\text{GA}}$: : sin. AGB : sin. ABG	
AGB=116° 40′	AB : $\underline{\text{GB}}$: : sin. AGB : sin. BAG	
ABG= 32 20	0·759668	0·759668
149 0	9·728227	9·711839
180	10·487895	10·471507
BAG= 31 0	9·951159	9·951159
	0·536736	0·520348
	∴ GA=3·44	∴ GB=3·31

[5.] To find GC.

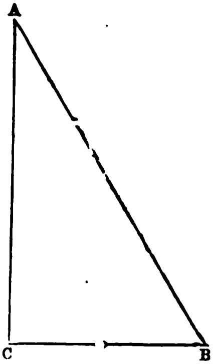
AC=8·25	AC : $\underline{\text{GC}}$: : sin. AGC : sin. CAG	
AGB=116° 40′	BAC=61° 46′	0·916454
BGC=112 30	BAG=31 0	9·708882
229 10	∴ CAG=30 46	10·625336
360		9·878875
∴AGC=130 50		0·746461
		∴ GC=5·58

Application of Plane Trigonometry to Navigation.

THE COMPASS.



It is proved in *Navigation*, Part II., that the course, distance, true difference of latitude, and departure between any two places on the surface of the earth, may be correctly represented by the sides and angles of a right-angled plane triangle. Let us suppose A and B to be the two places, and AC that part of the meridian passing through A that is intercepted between A and a straight line BC drawn through B perpendicular to AC. Then (see *Navigation*, art. 49) AC will represent the true difference of latitude, AB the distance, and BC the departure between A and B, and the angle CAB the course from A to B. It is manifest, that if in the triangle BAC any two of the above quantities are given, the other two may be found by the common rule for right-angled plane triangles. We will exemplify this by means of a few examples in sailing, and at the same time will

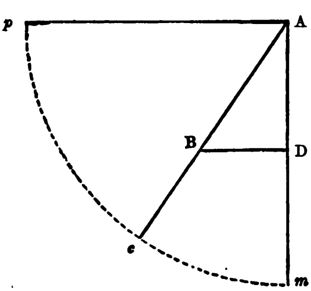


show how to find the ship's place by means of *construction*, that is, by the use of mathematical instruments; the practice thus obtained will be useful to the Naval student, and will form a proper introduction to the construction of charts and the tracing the ship's track thereon.

But before we can construct the ship's track for different courses, the several points of the compass must be thoroughly known. An expeditious method of forming the compass, and of learning it, is given in *Navigation*, Part I. p. 11.

10. A ship from latitude 47° 30′ N. has sailed S.W.b.S. 98 miles: find by construction, and by calculation, the latitude she is in, and the departure she has made.

(1.) *By Construction.*



Let A represent the point the ship departed from, AD the meridian, and Ap, drawn at right angles to it, the parallel of latitude of the ship. At the point A, with the chord of 60°, describe the quadrant mp, and cut off mc=S.W.b.S. or 33° 45′= the course; and through c draw a line AB. From a scale of equal parts take AB=98 miles, the distance; and through B draw BD parallel to Ap, meeting AD in D. Then B is the place the ship has arrived at, AD is the difference of latitude, and BD is the departure. If AD and BD are measured by the same scale of equal parts, it will be found that the difference of latitude AD is about 81 miles, and the departure BD about 54 miles. The figure may be more easily laid off by means of a protractor (see any work on Practical Geometry).

(2.) *By Trigonometry.*

In the right-angled triangle ABD are given the course DAB=33° 45′, and distance AB=98 miles; to find the difference of latitude AD, and departure BD.

By Rule, p. 57, $\frac{AD}{AB}$=cos. DAB ∴ AD=AB cos. DAB

By same Rule, $\frac{BD}{AB}$=sin. DAB ∴ BD=AB sin. DAB

Reducing these formulæ to logarithms, we have:

log. AD=log. AB+log. cos. DAB−10

log. BD=log. AB+log. sin. DAB−10

AB=98 DAB=33° 45′

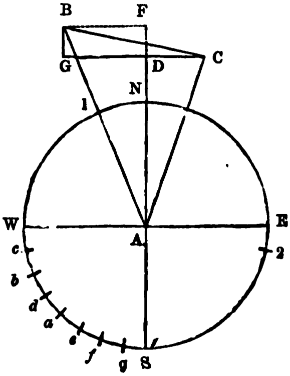
log. AB	1·991226	log. AB	1·991226
„ cos. DAB ...	9·919846	„ sin. DAB ...	9·744739
„ AD	1·911072	„ BD	1·735965
∴ AD=81·5=	1° 21′ 30″ S.	∴ BD=54·4	
Lat. from ...	47 30 0 N.		
Lat. in	46 8 30 N. and dep. 54·4 W.		

(2.) A ship from latitude 34° 40′ N. has sailed N.E.b.N. 196 miles: find latitude in, and departure (by Construction and by Logarithms). *Ans.* lat. in 37° 23′ N.; dep. 109′.

When a ship has described more than one course during the day, and it is required to show by a diagram the latitude she has arrived at, we may proceed as in the following example.

11. A ship in latitude 47° 30′ N. has sailed during the day N.N.W. 90 miles, and E.b.S. 60 miles: what latitude is she in, and what departure has she made?

(1). *By Construction.*



Let A represent the place the ship left, and with any convenient radius describe the circle NWSE to represent the horizon of the ship. Draw two diameters NS and WE, at right angles to each other. Let NS represent the meridian, and WE the parallel of latitude the ship departed from. To mark off the several courses and distances during the day, we may proceed as follows.

[1.] Divide one of the quadrants, WS, into eight equal parts to form a scale of points. This may be done by bisecting WS in *a*, and then W*a* in *b*, and W*b* in *c*; the other points, *d e f g*, may then be readily filled in.

[2.] Mark off on the circumference the several courses, thus: take N_1=N.N.W., or two points from the scale in WS; and E_2=E.b.S., or one point from the east.

[3]. Through A_1 draw the straight line AB, and make AB=90 miles by a scale of equal parts; and through B, parallel to a line passing through A_2, draw BC=60 miles. The point C represents the place the ship has arrived at. Join AC, and through C draw CD parallel to WE, meeting AN produced in D. Then AD is the difference of latitude, and DC the departure made good during the day. Also the angle CAD and line AC represent the direct course and distance from A to C.

If we measure AD by the scale of equal parts, we shall find the difference of latitude AD about 71 miles to the north, and the departure DC about 24 miles to the east of the place the ship left. The latitude arrived at is found thus:

Lat. A	47° 30′	N.
Diff. lat.........	1 11	N.
Lat. in	48 41	N. and dep. 24 E.

(2.) *By Trigonometry.*

To calculate the difference of latitude and departure between A and C, we must proceed as follows:

Through B draw BF parallel to WE, meeting the meridian produced in F. Then in the right-angled triangle ABF are given the course BAF=2 points, and distance AB=90 miles, to calculate AF the diff. lat. and BF the departure. Again, through B draw BG parallel to the meridian NS; and through C draw CG parallel to WE, meeting BG in G. Then in the triangle BGC are given the course GBC=7 points, and BC=60 miles, to calculate BG=FD the diff. lat. and GC the departure. By performing the calculation, we find that AF=83·2 miles to the north, and BG=11·7 to the south; so that the diff. lat.=83·2 N.—11·7 S.=71·5 miles. Similarly may be found GC=58·8 to the east, and FB or DG=34·4 to the west; so that the departure=58·8 E.—34·4 W.=24·4 E.

This method of computing the diff. lat. and departure separately for every course is in practice avoided by making use of a table called the Traverse Table, which contains the diff. lat. and departure, calculated for any given course and distance, so that these quantities may be found by inspection. The diff. lat. and departure, when taken out of the table, are arranged under proper heads in the following form :

Points.	Courses.	Dist.	Diff. lat.		Dep.	
			N.	S.	E.	W.
2	N.N.W.	90	83·2	—	—	34·4
7	E.b.S.	60	—	11·7	58·8	—
			83·2 11·7	11·7	58·8 34·4	34·4
			71·5 N.		24·4 E.	

Lat. from ...... 47° 30′ N.
Diff. lat. ...... 1 11·5 N.=71·5′ N.
Lat. in .... 48 41·5 N. and dep. 24·4 E.

(2.) A ship in latitude 32° 14′ N. has sailed during the day S.S.W. 45 miles, and W.b.N. 30 miles. Find latitude in, and departure. *Ans.* lat. in 31° 38′ N.; dep. 46·6.

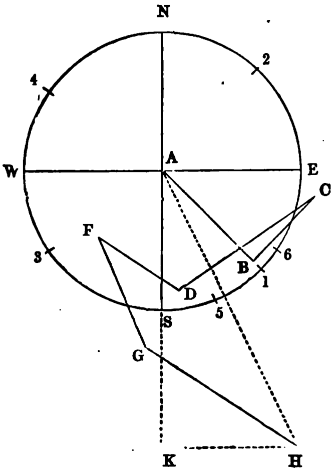
12. A ship from latitude 50° 48′ N. has sailed during the day on the following courses. Required the latitude in, and departure, and thence her direct course and distance from one place to the other.

1.	S.E.	40 miles.	4.	N.W.b.W. ...	30 miles.
2.	N.E.	28 „	5.	S.S.E.	36 „
3.	S.W.b.W. ...	52 „	6.	S.E.b.E.	58 „

(1.) *By Construction.*

Let A be the place sailed from, and NWSE the horizon of the ship. Draw the meridian NS, and parallel of latitude WE.

Divide one of the quadrants into eight equal parts for a scale of points, as in the last example, and by means of this scale mark off on the circumference the several courses, viz. S_1=S.E., N_2=N.E., S_3=S.W.b.W., N_4=N.W.b.W., S_5=S.S.E., and S_6=S.E.b.E.



Through A draw AB=40 by a scale of equal parts; through B, and parallel to A_2, draw BC=28 miles; through C, and parallel to A_3, draw CD=52 miles; through D, and parallel to A_4, draw DF =30 miles; through F, and parallel to A_5, draw FG=36 miles; and lastly, through G, and parallel to A_6, draw GH=58 miles. The point H is the place the ship has arrived at. Join AH, and through H draw HK parallel to WE and meeting the meridian NS produced in K. Then AK is the difference of latitude, and KH the departure made good during the day. Also the angle KAH represents the direct course, and the line AH the direct distance from A to H.

If we measure AK by the scale of equal parts, we shall find the difference of latitude AK about 86 miles to the south, and the departure KH about 42 miles to the east of the place the ship left. The latitude arrived at is found thus:

Lat. A .... 50° 48′ N.
Diff. lat......... 1 26 S.
Lat. in .... 49 22 N. and dep. 42′ E.

(2.) *By Trigonometry.*

The diff. lat. and departure for each course and distance may be computed as in Example p. 99. But to avoid this tedious operation, the several quantities may be taken out of the Traverse Table by inspection, as follows:

Points.	Courses.	Dist.	Diff. lat.		Dep.	
			N.	S.	E.	W.
4	S.E.	40	—	28·3	28·3	—
4	N.E.	28	19·8	—	19·8	—
5	S.W.b.W.	52	—	28·9	—	43·2
5	N.W.b.W.	30	16·7	—	—	24·9
2	S.S.E.	36	—	33·3	13·8	—
5	S.E.b.E.	58	—	32·2	48·2	—
			36·5	122·7 36·5	110·1 68·1	68·1
			Diff. lat.=86·2		42·0=dep.	

Lat. from......... 50° 48′ 0″ N.
Diff. lat. .... 1 26 12 S.=86·2 S.
Lat. in............. 49 21 48 N. and dep. 42′ E.

To calculate the direct course from A to H, or the angle KAH, and the direct distance AH, we have in the plane right-angled triangle AKH, AK = 86·2, and KH = 42; to find the course KAH, and distance AH.

[1.] To find the course KÀH.

$$\text{tan. KAH}=\frac{\text{KH}}{\text{AK}}$$

$$\therefore \text{log. tan. KAH}-10=\text{log. KH}-\text{log. AK}$$

$$\text{or log. tan. KAH}=10+\text{log. KH}-\text{log. AK}$$

KH=42 log. KH+10.. 11·623249
AK=86·2 „ AK......... 1·935507
„ tan. KAH 9·687742
∴ the course, or KAH=S.25° 58′ 30″E.

(See Practical Rule in *Navigation*, Part I. p. 44).

[2.] To find the distance AH.

$$\frac{\text{AH}}{\text{AK}}=\text{sec. KAH, or AH}=\text{AK sec. KAH}$$

$$\therefore \text{log. AH}=\text{log. AK}+\text{log. sec. KAH}-10$$

log. AK.................. 1·935507
„ sec. KAH−10 ... 0·046247
„ AH.................. 1·981754
∴ distance, or AH=95·8 miles.

(2.) A ship from latitude 50° 48′ S. has sailed during the day on the following courses:

1.	N.W.	20 miles.	4.	S.E.b.E......	15 miles.
2.	S.W.	14 „	5.	N.N.W.	18 „
3.	N.E.b.E. ...	26 „	6.	N.W.b.W. .	29 „

Required latitude in, and departure; and the direct course and distance from one place to the other.

Ans. Latitude in 50° 5′S.; departure 20·9 miles; course, N. 25° 55′ W.; dist. 47·8 miles.

The two following examples, taken out of that valuable old work, Robertson's *Elements of Navigation*, are given for practice in construction, and for the singularity of the form of the diagrams resulting from the several courses and distances.

13. A ship sails from a place in latitude 40° N. on the following courses. Required the latitude arrived at (by Construction).

Course.			Course.		
1. ...	S.E.b.S.	29′	13. ...	West	62′
2. ...	N.N.E.	10	14. ...	North.........	10
3. ...	E.S.E..........	50	15. ...	West	8
4. ...	E.N.E.	50	16. ...	South.........	10
5. ...	S.S.E..........	10	17. ...	West	62
6. ...	N.E.b.N.	29	18. ...	South.........	7
7. ...	West	25	19. ...	E.¾S.	62
8. ...	S.S.E..........	10	20. ...	South.........	110
9. ...	W.S.W.½W. .	42	21. ...	W.N.W.½W.	42
10. ...	North.........	110	22. ...	N.N.E.	10
11. ...	E.¾N..........	62	23. ...	West	25
12. ...	North.........	7			

Ans. The ship returns to the place sailed from.

14. Two ships, A and B, part company in lat. 31° 31′N., and meet together again at the end of two days, having run as follows:

The ship A.			The ship B.		
Course.			Course.		
1.	N.N.E.	96′	1.	N.N.W.	96′
2.	W.S.W.	96	2.	E.S.E.	96
3.	E.S.E.	96	3.	W.S.W.	96
4.	N.N.W.	96	4.	N.N.E.	96

Required the latitude arrived at, and the direct course and distance of each ship (by Construction).

Ans. Direct course due north, and distance 104 miles. Lat. in, 33° 15′N.

PROBLEMS IN SURVEYING, ASTRONOMY, AND NAVIGATION.

The following problems are given as Exercises in the preceding rules of Plane and Spherical Trigonometry. They are also designed to form an Introduction to Navigation and Marine Surveying.

[1]. (Fig. 17.)* On the opposite bank of a river to that on which I stood, is a tower known to be 216 feet high; with a pocket sextant I ascertained the angle between a horizontal line drawn from my eye (supposed to be 5 feet above the ground) and its top to be 47° 56′: required the distance across the river, from the place where I stood to the bottom of the tower. *Ans.* 190·4 feet.

[2]. (Fig. 53.) BDC is a straight line to which AD is perpendicular; AD is 100 feet high, and subtends an angle at B=36° 48′, and at C an angle=54° 30′: find the length of the line BC. *Ans.* 205 feet.

[3]. (Fig. 17.) A field is in the form of a right-angled triangle, whose base is 200 feet, and the angle at the base is 67°; how long will a man be walking round it at the rate of 4 miles an hour? *Ans.* 3 min. 21·6 sec.

[4]. (Fig. 53.) C and B are two places 100 feet apart, and A is a point equally distant from C and B; what must be the distance of the point A from C or B that the angle BAC may be 150°? *Ans.* 51·77 feet.

[5]. (Fig. 18.) A Maypole being broken off by the wind, its top struck the ground at 15 feet distance from the foot of the pole; required the height of the whole Maypole, supposing the length of the broken piece to be 39 feet. *Ans.* 75 feet.

---

* The figures or diagrams for the following problems are not drawn accurately to scale: they are given merely to indicate the form, without any regard to the value, of the sides and angles in the problems to which they refer.

[6]. (Fig. 19.) A ladder 36 feet long may be so placed that it shall reach a window 30·7 feet from the ground on one side of the street; and by only turning it over, without moving the foot out of its place, it will reach another window 18·9 feet high on the other side. What is the breadth of the street, and the angle of elevation of the first window from the second? *Ans.* Breadth of street 49·44 feet.
Angle of elevation 13° 25′.

[7]. (Fig. 17.) From the bottom of a tower, a distance AB=50 yards is measured on a horizontal plane, and at A, the angle BAC is found=25° 17′; required the height of the tower BC. *Ans.* 23·6 yards.

2*d*, When AB=12¼ and BAC=45°. „ 12¼ „

[8]. (Fig. 20.) To determine the distance of a ship at anchor at C, I measured a straight line AB=1000 yards on the shore, and observed the angles CAB=32° 10′ and CBA=83° 18′; required the distance of the ship from A.
Ans. 1100·1 yards.

2*d*, When AB=2·5 miles, CAB=80°, CBA=50°.
Ans, 2·5 miles.

[9]. (Fig. 20.) Two ships sailing in company, in order to determine nearly their distance from an object, C, on the shore, are separated from each other two nautical miles, AB; the angle is then observed from each ship between the object and the other ship; at A the angle is 85° 10′, at B it is 82° 45′; required the distance of each ship from the object.
Ans. 9·478, and 9·52 miles.

[10]. The base of an isosceles triangle is 120 feet, and the vertical angle 47°; find the sides.
Ans. Each side 150·5 feet.

[11]. (Fig. 17.) The two sides, AB and BC, of the right-angled triangle ABC are 18 and 24; find the length of the perpendicular drawn from the right angle upon the hypothenuse. *Ans.* 14·4.

[12]. (Fig. 21.) To determine the height, AB, of a tower inaccessible at the base, two stations, C and D, are chosen in

a horizontal plane, so that a plane passing through the three points, A, C, and D, is perpendicular to the horizon; the distance CD=100 yards, and the angle ACB=46° 15′, and BDA=31° 20′; required the height AB. *Ans.* 145·9 yards.

2*d*, When CD=300 yards, ACB=58°, and BDA=32°. *Ans.* 307·4 yards.

[13]. (Fig. 21.) From the decks of two ships at D and C, 880 yards asunder, the angle of elevation of a cloud at A on the same point of the compass from each is observed; at D the angle is 35°, at C it is 64°; required the height of the cloud above the surface of the sea, the deck of each ship being supposed to be elevated above it 21 feet. *Ans.* 942·6 yards.

[14]. A tower subtended 39° to an observer stationed 200 feet from the base: find the height, and also the angle it will subtend to an observer at 350 feet from the base.

Ans. Height, 162 feet; angle, 24° 50′.

[15]. (Fig. 20.) To determine nearly the distance between two ships at sea, I carefully observed the interval of time between the flash and report of a gun from each, and measured the angle which the two ships subtended. The intervals were 4 seconds and 6 seconds, and the angle observed =48° 42′; required the distance of the ships from each other (the velocity of sound being supposed to be 1142 feet in a second). *Ans.* Distance 5147·9 feet.

2*d*, When the intervals are 10 seconds and 5 seconds, and observed angle 60°. *Ans.* Dist. 9889·8 feet.

[16]. (Fig. 21′.) From B, the top of a ship's mast, which was 80 feet above the water, the angle of depression* of another ship's hull at C upon the water was 20°; required the distance of the ships. *Ans.* 219·8 feet.

---

* The angle between a horizontal line passing through the eye of the observer and a line drawn to the object is called the *angle of elevation* when the object is above the horizontal line, and the *angle of depression* when the object is below it; thus in fig. 21′, BCA is the angle of elevation of B above the horizontal line AC, and DBC is the angle of depression of C below the horizontal line BD.

2d, If the mast=143 feet, and the angle of depression =35°. *Ans.* 204·2 feet.

[17]. There are two monuments whose heights are 100 feet and 50 feet respectively: I observed, with a sextant, that the line joining their tops when produced made with the horizontal plane an angle=37°; required their distance apart.
Ans. 66·4 feet.

[18]. (Fig. 23.) Wishing to determine the height of an obelisk standing on the top of a declivity, I measured from its base a distance of 40 feet, and then observed the angle formed by this line and a line drawn to the top to be=41°. Going on in the same direction 60 feet farther, the angle formed by the declivity with a line drawn to the top was =23° 45′. Required the height of the obelisk.
Ans. Height, 57·62 feet.

[19]. (Fig. 24.) To determine the distance between two inaccessible objects, C and D, as two ships at anchor, a base, AB, on the same plane as the objects was measured and found to be 670 yards; the following angles were also observed at the extremities of the base: at A, BAD=40° 16′, BAC=97° 56′; at B, ABC=42° 22′, and ABD=113° 29′; required CD. *Ans.* CD=1174·4 yards.

[20]. (Fig. 24.) To determine nearly the distance of two redoubts, C and D, by which the entrance into a harbour is defended, a boat is placed at A with its head towards a distant object seen at E, and the angles CAD=22° 17′, and DAE=48° 1′ were observed. The boat is then moved to B, a distance of 1000 yards, directly towards E, and the angles CBD=53° 15′, and DBE=75° 43′ are observed; required the distance CD. *Ans.* Distance, 1290 yards.

[21]. A right-angled triangle rests on its hypothenuse, the length of which is 100 feet; one of the angles is 36° 40′, and the inclination of the triangle to the horizon is 60°; find the height of the right angle above the ground.
Ans. 41·48 feet.

[22]. (Fig. 25.) To determine the height of an object, EB, on the top of an inaccessible hill, I took the angle of elevation, ACE, of the top of the hill=40°, and also that of the top of the object ACB=51°. Going then 100 yards in a direct line from the object and upon a horizontal plane, I found the angle of elevation of the top of the object ADB =33° 45′; required the height of the object.

Ans. Height, 46·67 yards.

[23]. (Fig. 26.) Wanting to know the distance between two objects, A and B, which could only be seen from a particular place, D, I set up two staffs, at C and E, and took the angles ADC=89°, ADB=72 30′, and BDE=54° 30′. I then measured DE and DC, each=200 yards, and took the angles BED=88° 30′ and DCA=50° 30′; required the distance AB. *Ans.* 345·5 yards.

[24]. (Fig. 27.) To determine my distance from an inaccessible object at O, *without observing any angles*, I measured a straight line AB=500 yards, from each extremity of which I could see O. I then measured from A and B, in a direct line from O, AC and BD, each=100 yards; finally, I measured the diagonals AD and BC, the former was=550 yards, the latter=560 yards; required the distance of the object from A and B. *Ans.* AO=536·2, BO=500·47 yds.

[25]. (Fig. 28.) Wanting to know the distance, AC, of a hill from the station, A, and also its height, OC, I measured a base, AB=298 yards, on ground nearly level, and at A and B observed with a sextant the angles BAO=42° 17′ and ABO=79° 29′; and at A, the angle of elevation OAC=4° 51′; required the distance AC and OC.

Ans. AC=343·4, OC=29·14.

[26]. (Fig. 29.) Find how much the point B is elevated above a point H, from the following data: having observed the angle of elevation BAH=4° 38′, and measured AC=193 yards to a point C, at C the angle ACB=76° 32′, and at A the angle BAC=45° 4′. *Ans.* BH=17·8 yards.

[27]. (Fig. 29.) The angle of elevation of a tower 100 feet high, due north of an observer, was 50°; what will be its angle of elevation after walking due east 300 feet?

Ans. 17° 47′ 45″.

[28]. (Fig. 30.) The elevation of a balloon was observed to be 20° bearing N.E., and by another observer 4000 yards due south of the former it was N.b.E.; required its height.

Ans. 511·3 yards.

[29]. (Fig. 31.) From a window which seemed to be on a level with the bottom of a steeple, I observed the angle ACB of elevation of the top of the steeple=40°; from another window of the same house, 18 feet directly above the former, the angle EDB of elevation was=37° 30′; required the height and distance of the steeple.

Ans. Height, 210·4; distance, 250·8 feet.

[30]. (Fig. 32.) At B, the top of a castle which stood on a hill near the sea-shore, the angle of depression, HBS, of a ship at anchor was 4° 52′, and at R, the bottom of the castle, its depression, NRS, was 4° 2′; required the height of the top of the building above the level of the sea, supposing the castle itself to be 54 feet high; required also the horizontal distance of the vessel. *Ans.* Height, 314·2; distance, 3690·3 feet.

[31]. (Fig. 33.) Wishing to know the breadth of a river, I measured a base of 500 yards in a straight line close to one side of it, and at each extremity of the base I observed the angles subtended by the other end and a tree standing on the opposite bank to be 53° and 79° 12′; required the breadth of the river. *Ans.* Breadth, 529·5 yards.

2d. If base=108·3 and angles 43° 12′ and 60° 35′.

Ans. Breadth, 66·49.

[32]. (Fig. 34.) A base of 340 feet was measured on a sloping side of a hill, in a vertical plane passing through its summit. It made an angle of 10° 5′ with the horizontal plane; at the higher end of the base, the elevation of the hill was 46° 15′, and at the other end it was 40° 10′. The height

of the eye at each observation was 5 feet: find the height of the hill above the lower end of the base. *Ans.* 1226 ft.

[33]. (Fig. 35). The distance between two stations B and C on a declivity is 220 yards. At B, the *horizontal* angle C*ba*, between C and an object A on the top of a hill, was found by a *theodolite* to be 70° 15′, and at C, the horizontal angle *b*C*a*, between B and A, was 62° 33′; the vertical angles $ACa=32° 12'$ and $BCb=8° 32'$; find the horizontal distances of the object A from C and B, namely C*a*, and *ba*, and also the heights A*a* and AD of the object above C and B.

Ans. Hor. dist. of A from C= $Ca=279{\cdot}1$ yards.
„ „ B= $ba=263{\cdot}1$ „
perp. height of A above C= $Aa=175{\cdot}7$ „
„ „ B=$AD=143{\cdot}1$ „

[34.] (Fig. 35). Suppose, as in the last problem, that the direct distance between the stations B and C is 220 yards, also that at C the angle BC*b* (the elevation of B)=8° 32′ and AC*a* (the elevation of A)=32° 12′; but suppose now that a *sextant* is used in measuring the oblique angles, and that $ABC=77° 8'$ and $ACB=62° 18'$. Find, as before, the *horizontal* distances between A and the stations B, C, and the heights of A above B and C. *Ans.* See last problem.

[35]. (Fig. 36.) Wanting to know the distance of two objects, A and B, from each other, and from another object, D, all in the same plane, on BA, produced on the side of A, a point C was taken, and CD being measured was found to be 549·4 yards, and the angle C=57°. At D the angle CDA was observed to be 14°, and the angle BDA=41° 30′; required the distance of A, B, and D from each other.

Ans. $AB=349{\cdot}52$, $AD=487{\cdot}27$, and $BD=498{\cdot}7$ yards.

[36]. (Fig. 37.) Wishing to find the distance of a battery at B from a fort at F, which cannot be seen from the battery in consequence of the ground between B and F being covered with wood, &c., I measured distances, BA and AC, to points A and C, where both the fort and battery were visible, the former being 2000 yards and the latter 3000;

and observed the angles BAF=34° 10′, FAC=74° 42′, and FCA=80° 10′. From these data it is required to find the distance of the fort from the battery. *Ans.* 5422 yards.

[37]. (Fig. 38.) Coming from sea, at the point D, I observed two headlands A and B, and inland at C a steeple which appeared between the headlands: I found from a map that the headlands were 5·35 miles from each other; that the distance from A to the steeple was 2·8 miles, and from B to the steeple 3·47 miles. I observed with a sextant the angle ADC=12° 15′ and BDC=15° 30′; required my distance from each of the three objects. (See Example, p. 91.)

Ans. AD=11·26, CD=12·46, BD=11·03 miles.

[38]. (Fig. 39.) Required the distance of the three objects A, B, and C from the point D, situated *within* the triangle, from the following data: AB=267, AC=346, BC=209, angles ADC=128° 40′, and ADB=91° 20′. (See Example, p. 93.) *Ans.* AD=249, BD=91, CD=131.

The following twelve problems require a knowledge of the compass (p. 97).

[39]. (Fig. 40.) A headland C bore due north of a ship at A; after sailing 10 miles due east to B, the headland bore N.W.; required the distance of the headland at each observation. *Ans.* 10 and 14·14 miles.

[40]. (Fig. 41.) A fort A bore from a ship C due north, by compass, and after sailing N.E. by compass 14½ miles to B, the fort bore N.W.; find the distance of the fort at both observations. *Ans.* 20·5 and 14·5 miles.

[41]. (Fig. 42.) A boat is placed at A due west of a ship at B; after sailing N.W. 10 miles to C, the boat bears S.S.W.; required the distance of the boat from the ship at the first observation. *Ans.* 10 miles.

[42]. (Fig. 43.) Sailing along a coast, a headland C was observed to bear N.E.b.N.; having run E.b.N. 15 miles to B, the headland bore W.N.W.; find the distance from the headland at each observation. *Ans.* 8·496, 10·81 miles.

[43]. A and B are two points lying north and south 50

yards apart; what must be the distance of a third point C from each, that it may bear N.b.W. from B, and W.b.S. from A? *Ans.* 9·75 and 49·04 yards.

[44]. (Fig. 44.) A cape C was observed to bear from us N.W., and another headland H to bear N.N.E.½E.; standing away E.N.E.½E. 23 miles to B, we found the first bore from us W.N.W., and the second N.b.W.½W.; required the distance and bearing of the cape from the headland.

Ans. S. 5¼W., 42·33 miles.

[45]. A ship sailing N.W., two islands appeared in sight, one bearing W.N.W., the other N. from the ship, and when the ship had sailed six miles farther, the first bore W.b.S., and the other N.E.; required their bearing and distance from each other. *Ans.* S. 58° 40′W., distance 9·71 miles.

[46]. From two stations on the deck of a ship 100 feet apart, the bearings of an object on shore were N.E. and N.N.E., and the ship's head was N.b.W.; find the distances of the object from each station. *Ans.* 145·2 and 217·3 feet.

[47]. (Fig. 45.) A church C bears from a battery B, E.N.E. 960 yards; how must the church bear from a ship at sea, supposing her to run in until the battery is north 2000 yards? *Ans.* N. 20° 32′E.

[48]. (Fig. 46.) A cape C bears from a headland H, W.¼S. 4·23 miles; how must the cape bear from a ship which runs in towards the headland on a N.b.W.½W. course, until the headland is 2·3 miles distant from the ship?

Ans. W.N.W.

[49]. (Fig. 47.) A ship S was 2640 yards due south of a lighthouse AB, and after sailing N.W.b.N. 800 yards to D, its angle of elevation BDA was 5° 25′; required its height.

Ans. 192 yards.

[50]. (Fig. 48.) The bearings of two objects A and B in the same latitude from a ship at C are N.N.E. and N.E.b.E., and the distance from A is 10 miles; required the distance from B. *Ans.* 16·63 miles.

[51]. (Fig. 49.) Find the angle which the *line of metal*

makes with the axis produced in a piece whose dimensions are as follows: diameter of breach=12·44 inches; of muzzle =9·84 inches; length of gun=77¼ inches. *Ans.* 0° 57′ 51″.

Calculations of a front of Fortification.

[52]. (Fig. 50.) Let AB be the exterior side of a regular pentagon, and suppose the side to be 350 yards, the perpendicular DE 50 yards, the face AH of the bastion 100 yards, and the line of defence AG equal to AK. As the length and position of the other lines depend on these values, it is required to find the magnitude and position of the following parts:

1. Angle of the tenaille AEB.	*Ans.*	148° 6′ 30″.
2. Line of defence AK or AG.	,,	255·33 yards.
3. Flanked angle HAH_1	,,	76° 6′ 30″.
4. Curtain FG.	,,	141·02 yards.
5. Angle of the shoulder AHF.	,,	117° 0′.
6. Length of flank FH.	,,	43·5 yards.
7. Flanked angle GFH.	,,	101° 3′.

The following problems are given as exercises in analytical trigonometry and algebra.

[53*a*]. What angle will a tower subtend at a place whose distance is equal to six times its height? and where must an observer station himself that the angle of elevation may be double of the former? *Ans.* 9° 28′; 2·9 times the height.

[54*a*]. (Fig. 51.) A maypole was broken by the wind, and its top struck the ground 20 feet from its base; and being again fixed was broken a second time 5 feet lower, and its top extended 10 feet farther; required its height. *Ans.* 50 ft.

[55*a*]. (Fig. 52.) The summit A of a hill bore due east of a spectator at B, and E.N.E. of a spectator at a point C due south of B; the elevation of the point A at B was 20°; required its elevation at C. *Ans.* 18° 36′.

[56]. (Fig. 53.) If the base of an oblique-angled triangle

be 40, and the other two sides 20 and 30, what is the length of the perpendicular from the vertical angle? *Ans.* 14·523.

[57]. (Fig. 54.) The elevation of a spire DC at one station A was 23° 50′ 15″, and the horizontal angle at this station, between the spire and another station B, was 93° 4′ 20″; the horizontal angle at B was 54° 28′ 30″, and the distance AB between the two stations was 416 feet; required the height of the spire. *Ans.* 278·7 feet.

[58*a*]. (Fig. 52.) An observer finds the angle of elevation of a tower at a point B to be 23° 18′; after walking from B 300 feet in a direction at right angles to the line joining B with the foot of the tower, the angle of elevation was 21° 16′; required the height of the tower, and its distance from B.
Ans. Height, 272·7 feet; distance, 633·4.

[59.] A pole is inclined at an angle of 80° to the horizontal plane, and when standing 100 feet from it in the direction in which it is inclined, the angle of elevation of the top is 54°; required its length. *Ans.* 112·5 feet.

[60*a*]. (Fig. 55.) On the bank of a river stands a column 200 feet high, on which is a statue 50 feet high, and to an observer on the opposite bank the statue subtended an equal angle with a man 6 feet high standing at the base of the column; required the width of the river. *Ans.* 82·5 feet.

[61*a*]. (Fig. 24.) The distance between two objects, C and D, is known to be 6594 yards; on one side of the line CD there are two stations, A and B, at which angles are taken. The angle CAD=85° 46′, DAB=23° 56′, CBD=68° 2′, and CBA=31° 48′. From these observations it is required to find the distance between the stations A and B.
Ans. AB=4694 yards.

[62*a*]. (Fig. 56.) A flag-staff, 12 feet high, on the top of a tower, subtended an angle of 48′ 20″ to an observer at the distance of 100 yards from the foot of the tower; required the height of the tower. *Ans.* 401·4 feet.

[63]. Find the area of a triangle whose base is 40 feet, and perpendicular 30 feet. *Ans.* 66⅔ square yards.

[64*a*]. If from a right-angled triangle whose base is 12 and perpendicular 16 feet, a line be drawn parallel to the perpendicular, cutting off a triangle whose area is 24 feet, required the sides of the triangle. *Ans.* 6, 8, 10 feet.

[65*a*]. Required the side of an equilateral triangle, the area of which is 180 square yards. *Ans.* 20·389 yards.

[66*a*]. Given the base of a triangle equal to 476·25 yards, and the angles at the base 27° 10′ 15″ and 35° 10′ 15″; to find the area. *Ans.* 33680.

[67*a*]. The area of a triangle is 6, and two of its sides are 3 and 5; find the third side. *Ans.* 4 or $\sqrt{52}$.

[68*a*]. (Fig. 57.) The straight line EF is drawn parallel to the base of the triangle ABC, whose altitude is 10 feet; find the distance of EF from the base BC, so that it may divide the triangle into two equal parts.

Ans. Distance from base, 2·929 feet.

[69*a*]. When a parish was enclosed, the allotment of one of the proprietors consisted of two pieces of ground, one of which was in the form of a right-angled triangle; the other was a rectangle, one of the sides of which was equal to the hypothenuse of the triangle, the other to half the greater side: but wishing to have his land in one piece, he exchanged his allotments for a square piece of ground of equal area, one side of which equalled the greater of the sides of the triangle which contained the right angle. By the exchange, he found he had saved 55 yards of paling. Required the area of the triangle and rectangle. *Ans.* Triangle, 181·5; rectangle, 302·5.

[70*a*]. The area of a right-angled triangle whose sides are in arithmetical progression = 216; determine the sides.

Ans. 18, 24, 30.

[71*a*]. What is the side of that equilateral triangle whose area cost as much paving, at 8*d*. per foot, as the palisading the three sides did at 7*s*. a foot? *Ans.* 72·74 feet.

[72*a*]. Given a side, a, of a regular polygon of n sides, to find a formula for computing the area.

Ans. $\text{Area}=\frac{na^2}{4}\cot.\frac{180^\circ}{n}$.

EXAMPLES.

1. An eight-sided polygon, or octagon, whose side=16 yards. *Ans.* Area=1236·1.

2. A ten-sided polygon, or decagon, whose side=20½ yards. *Ans.* Area=3233·5.

[73*a*]. Given the area A of a regular polygon of n sides, to find a side a.

Ans. $a=\sqrt{\frac{4\text{A}}{n}\tan.\frac{180^\circ}{n}}$.

EXAMPLES.

1. The area of a regular octagon is 1236·1 square yards; find a side. *Ans.* Side 16 yards.

2. The area of a decagon is 3233·5 square yards; find a side. *Ans.* Side 20·5 yards.

[74*a*]. (Fig. 58.) To make a regular polygon of n sides equal to a given triangle ABC. Take $\text{AD}=\frac{\text{AB}}{n}$; draw AG, making angle $\text{CAG}=\frac{360^\circ}{n}$; draw DE parallel to AC; and take AF=a mean proportional to AC and AE; then AF is the radius of a circle that will contain the required polygon: required a proof.

[75]. A ship sailing on a S.S.W. course, bore from me due south, and the angle subtended by the head and stern was 20′ 15″, and her length was known to be 160 feet; required her distance. *Ans.* 1·94 miles.

[76*a*]. A ship sailed S.x°W., and met another ship which had sailed N.$(x+10^\circ)$W. from the same meridian; the distances sailed were as 3 : 2, and their distance from the meridian left was 100 miles; required the difference of latitude.

Ans. 479·8 miles.

[77*a*]. How far may the surface of the sea be seen by a man standing 6 feet above it, the radius of the earth being 4000 miles? *Ans.* 5307 yards.

[78*a*]. From the top of a mountain 3 miles high, the true depression of the horizon was found to be 2° 13′ 27″; required the diameter of the earth, supposing it to be a sphere.

Ans. 7952 miles.

[79*a*]. If at the top of a mountain the true depression of the horizon of the sea is found to be 1° 31′; what is the height of the mountain, supposing the earth to be a sphere whose diameter is 8000 miles? *Ans.* 1·402 miles.

[80*a*]. From a station on the side of a river whose banks are parallel, and width 560 yards, the angle formed by two objects on the opposite bank was 35° 10′, and their distance from each other 400 yards; required their distance from the station. *Ans.* 560·1 and 694·3 yards.

[81*a*]. Walking along a road, I observed the elevation of a tower, AB, to be 20°, and the angular distance of its top from an object in the road to be 30°, the nearest distance of the tower from the road was 200 feet; required its height.

Ans. 187·5 feet.

[82*a*]. One angle of a triangle is 129° 34′, and the two sides about that angle are to each other in the proportion of 4 to 7; required the other two angles.

Ans. 32° 32′ and 17° 54′.

[83*a*]. The three sides of a plane triangle $=6$, and the angles are to each other as 1, 2, 3; find the sides.

Ans. 1·268, 2·536, 2·196.

[84*a*]. The perimeter of a triangle is equal to 100 yards, and the angles are to each other in the proportion of 1, 2, and 4; it is required to find the sides of the triangle.

Ans. 19·8, 35·69, 44·51 yards.

[85*a*]. The perimeter of a right-angled triangle $=24$ yards, and one of the angles $=30°$; find the sides.

Ans. 5·072, 8·784, 10·144 yards.

[86*a*]. In a plane triangle ABC, given $A=80^\circ$, and $a=400$; required the other sides, their sum being 600.

Ans. 369·25, 230·75.

[87*a*]. In a plane triangle ABC, given $A=60^\circ$, $c=400$, and the sum of the other two sides, a and b, $=600$; required the sides a and b. *Ans.* 250 and 350.

[88*a*]. The perimeter of a right-angled triangle is 24 feet, and its base is 8 feet; find the other sides.

Ans. 6 and 10 feet.

[89*a*]. At 80 feet distance from a steeple, the angle made by a line drawn from its top to the place was double that made by a line drawn from the top to a point 250 feet from the steeple on the same level; required the height of the steeple. *Ans.* 150 feet.

[90*a*]. Given the base a, the vertical angle A, and the sum of the other two sides of the plane triangle $\text{ABC}=b$; to find the sides x and y.

Ans. $x+y=b$, and $xy=\frac{1}{4}(b+a).(b-a)\ \sec.^2\frac{A}{2}$ from which equations x and y may be found.

[91*a*]. Given the base a, the vertical angle A, and the difference of the sides of a plane triangle$=d$, to find the sides x and y.

Ans. $x-y=d$, and $xy=\frac{1}{4}(a+d).(a-d)\ \text{cosec.}^2\frac{A}{2}$

[92*a*]. Given the base a, the difference of the angles x and y at the base$=d$, and the sum of the two other sides of the plane triangle$=b$, to solve the triangle.

Ans. $\text{Cos.}\ \frac{1}{2}(x+y)=\frac{a}{b}\cos.\frac{1}{2}d$, which determines the sum of the unknown angles, and thence, with their difference already known, the angles x and y.

[93*a*]. Given the base a, the difference of the angles x and y at the base$=$D, and the difference of the sides of a plane triangle$=d$, to solve the triangle.

Ans. $\text{Sin.}\ \frac{1}{2}(x+y)=\frac{a}{d}\sin.\frac{1}{2}\text{D}$, and $x-y$ is already known.

[94*a*]. Given the base of a triangle b, and one of the angles at the base A, and the difference of the other sides$=d$, to solve the triangle.

Ans. Tan. $\frac{1}{2}\mathrm{C}=\frac{b-d}{b+d}\tan.\frac{1}{2}\mathrm{A}$, from which the angle C is found, and thence the other parts of the triangle.

[95*a*]. Given the base of a plane triangle b, one of the angles at the base A, and the sum of the other sides$=m$, to solve the triangle.

Ans. Tan. $\frac{1}{2}\mathrm{C}=\frac{m-b}{m+b}\cot.\frac{1}{2}\mathrm{A}$.

[96*a*]. Given the angles and the perimeter of a plane triangle, to find the sides.

EXAMPLE.

Perimeter$=100$ yards, $\mathrm{A}=102°\,51'\,30''$, $\mathrm{B}=25°\,42'\,45''$, and $\mathrm{C}=51°\,25'\,45''$. *Ans.* $a=44{\cdot}51$, $b=19{\cdot}8$, $c=35{\cdot}69$ yards.

[97*a*]. Given the distances between three stations in a straight line with an object standing upon a horizontal plane; and the angles at the points E, D, C,$=\theta$, $90-\theta$, and 2θ in order, θ being unknown, to find its height.

EXAMPLE.

Let $\mathrm{ED}=20$, and $\mathrm{DC}=20$. *Ans.* 38·73.

[98*a*]. Investigate analytical expressions for calculating the distance of a station from each of three points, having given the distances of the points from each other, and the angles which they subtend at the station. For numerical examples, see Probb. 37, 38.

[99*a*]. If in the three edges which meet at one angle of a cube, three points A, B, C, be taken at distances a, b, c, from the angle respectively, the area of the triangle ABC, formed by joining the three points with each other, $=\frac{1}{2}\sqrt{a^2b^2+a^2c^2+b^2c^2}$.

[100*a*]. On the sides of an equilateral triangle three squares are described. Show that the area of the triangle formed by joining the centres of these squares$=$area equilateral triangle $\times(1+\frac{1}{2}\sqrt{3})$.

[101*a*]. A fleet of steam-vessels, under the command of an admiral, is steaming due east, 6 miles an hour; a part of the fleet under a vice-admiral bears from the admiral N.N.E. 6 miles, and a look-out steamer is S.E.b.S. 10 miles. The admiral, suspecting the enemy to be in the S.E. quarter, directs the vice-admiral, by signal, to take station 3 miles S.E. of him, and the look-out steamer due east 5 miles; at the same time he sends several steamers from the main body to reinforce the vice-admiral at the appointed station. Now, supposing the admiral continue to steer due east 6 miles an hour, and the vice-admiral to increase his speed to 10 miles, and the look-out steamer to 14 miles an hour, it is required to find the course and distance of the look-out steamer, and also the course and distance of the vice-admiral, so that they may reach their several stations in the shortest time. Required also the time in reaching their stations, the rate at which the reinforcement must steam to reach the station at the same time as the vice-admiral, and also its course and distance.

Ans. Vice-admiral's course is S.E.b.S. nearly; distance, 9·42 miles. Look-out steamer's course is N.N.E. nearly; distance, 8·94 miles. Reinforcement's course is E.b.S.½S. nearly; distance, 8 miles. The vice-admiral and reinforcement reach their station in 56 minutes, and the look-out steamer in 38 minutes. The reinforcement's rate of steaming, 8½ miles an hour.

If a geometrical construction of this problem be made on a large scale, the course, &c. so found will be sufficiently correct for practical purposes. (See a geometrical construction in the volume of Solutions.)

The above example belongs to a very important class of problems connected with naval tactics and the combined evolutions of steam squadrons. These problems will attract more attention hereafter, when it is found that a fleet of steamers must be handled with almost the same precision as an army.

SECTION II.

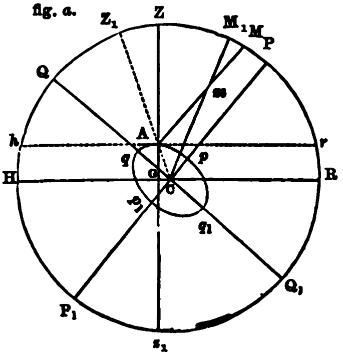
Application of Spherical Trigonometry to Nautical Astronomy.

(39). To assist the student in understanding the following problems, we will give in this place a few definitions of astronomical and nautical terms. These definitions are explained more at large in *Nautical Astronomy*, Part II.

Definitions in Astronomy.

(40). The astronomer conceives all the heavenly bodies to be contained within a hollow sphere of great but indefinite magnitude; he supposes this sphere (the interior surface of which is called the *celestial concave*) to have the same centre as the earth; and in all applications of spherical trigonometry, he employs, instead of the real position of any of the heavenly bodies, the point in which the celestial concave is cut by a straight line drawn through the *centres* of the earth and of the body; this point is called the *true* place of the heavenly body. Thus, let $QZPQ_1P_1$ represent the celestial concave, m a heavenly body, Apq_1p_1 the earth, and C its centre;* join Cm, and produce it to cut the celestial concave in M_1: then M_1 is the *true* place of the heavenly body m. Let A be the place of a spectator on the surface of the earth; join Am, and pro-

fig. a.



---

* The figure of the earth is proved to be an oblate spheroid (see *Navigation*, Part II.). In the diagram Apq_1p_1 is a section of the earth, but its elliptical form is much exaggerated, for the sake of drawing the lines distinctly.

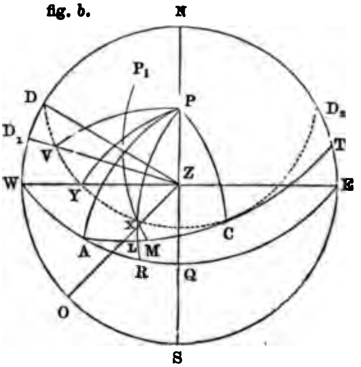
duce it to the celestial concave at M: then M is called the *apparent* place of the heavenly body *m*.

(41). The extremities of the axis on which the celestial concave appears to revolve, in consequence of the earth's diurnal motion about its axis pp_1, are called the *poles of the heavens*: thus the line PP_1 represents the axis of the heavens, P and P_1 are the poles; the points p and p_1 in which the earth's surface pqp_1q_1 is cut by this axis, are called the *poles of the earth*. That great circle on the surface of the earth which is equidistant from each of its poles is called the *terrestrial equator;* in the figure, qq_1 represents the plane of the terrestrial equator. The terrestrial equator extended to the celestial concave, as QQ_1, forms the *celestial equator*. A circle touching the earth where the spectator stands, and extending to the celestial concave, is called the *visible* horizon, and a circle parallel to the visible horizon which passes through the centre of the earth and extends to the celestial concave is called the *rational* horizon; thus *hr* represents the visible, and HR the rational horizon of the spectator at A. These two circles, however, form one and the same in the celestial concave: thus the points R and r in the figure must be supposed to coincide. This may be readily conceived, when we consider that the distance of any two points on the surface of the earth will make no sensible angle at the celestial concave; therefore either of these two circles is to be understood by the word *horizon*. Of the poles of the horizon of any place, that which is over the place is called the *zenith*, and the other the *nadir*, as Z and z_1 in fig. *a*.

(42). Great circles passing through the zenith are called *circles of altitude*, or *vertical circles;* the circle of altitude passing through the poles of the heavens is also called the *celestial meridian;* the points of the horizon through which the celestial meridian passes are called the north and south points; and a circle of altitude at right angles to the meridian is called the *prime vertical*. The points of the horizon through which the prime vertical passes are called the east and west points.

(43). These circles will be more clearly understood by means of fig. *b*, in which NWSE represents the horizon of a spectator whose zenith is supposed to be Z; N, S, E, W the north, south, east, and west points of the horizon. NZS is a circle of altitude passing through the zenith and pole, and therefore represents the celestial meridian: ZD, ZO, &c. are circles of altitude, or vertical circles, and WZE is the prime vertical (42).

fig. *b*.



(44). Since the horizon and celestial equator are both perpendicular to the celestial meridian, the points where the horizon and celestial equator intersect each other must be 90° distant from every part of the meridian [Part II. Art. (65), Cor. 3]; that is, *the celestial equator cuts the horizon in the east and west points;* draw the curve EQW to cut the horizon in the east and west points; this will represent the celestial equator. Since the poles of the heavens are 90° distant from the equator, take QP=90°. Then P is one of the poles of the heavens; it is called the elevated pole, or the one above the horizon, and the arc NP is the altitude of the pole.

(45). The *distance*, ZQ (figures *a* and *b*), of the zenith from the equator represents the *latitude of the spectator;* this may be more clearly seen in fig. *a*, p. 123, where PZQH represents the plane of the celestial meridian, Z the zenith of the spectator at A, *hr* or HR his horizon, P the elevated pole, QQ_1, drawn at right angles to P_1P, represents the plane of the celestial equator, and the angle ZGQ (the earth being considered as a spheroid) is the latitude of the spectator; but since the distance, ZQ, of the zenith from the equator measures the same angle, this arc is taken to represent the latitude.* ZP (figures *a* and *b*), the complement of ZQ, is called the *colatitude*.

(46). The altitude of the pole is easily shown to be equal to the latitude of the spectator;

$$\begin{aligned}\text{for alt. } NP &= 90^\circ - PZ = 90^\circ - \text{colat.} \qquad (\text{fig. } b).\\ &= 90^\circ - (90^\circ - \text{lat.}) = \text{latitude.}\end{aligned}$$

(47). The sun, in consequence of the earth's motion in its orbit, appears to move eastward in the celestial concave, and, in the course of a year, to describe among the fixed stars a great circle; this circle is called the *ecliptic*. It is inclined to the celestial equator at an angle of about 23° 28′, called the *obliquity* of the ecliptic. In fig. *b*, ACT represents a part of the ecliptic, and the angle MAR the obliquity. The points in which the equator and ecliptic intersect are called the *first point of Aries* and *first point of Libra;* the former, being the point where the sun crosses the equator to the northward, is called the *vernal equinoctial point*, and the latter the *autumnal* equinox. Great circles passing through the poles of the heavens are called circles of *declination*, and great circles passing through the poles of the ecliptic are circles of *latitude:* thus, in fig. *b*, PR, PA, &c. are circles of declination; and if P_1 represent the pole of the ecliptic, P_1M is a circle of latitude. *Parallels of declination* and *latitude* are small circles parallel respectively to the celestial equator and ecliptic.

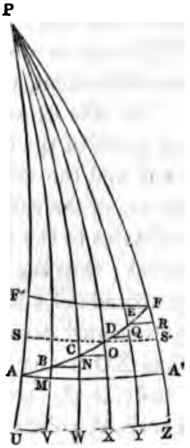
(48). The *declination* and *right ascension* of a heavenly body may be defined thus: the *declination* of a heavenly body is the arc of the circle of declination passing through its place in the celestial concave, intercepted between this place and the celestial equator; the *right ascension* of a heavenly body is the arc of the celestial equator intercepted between the first point of Aries and the circle of declination passing through the place of the body in the celestial concave. Or, it is the angle at the pole of the heavens between the circles of declination passing through the first point of Aries and the place of the heavenly body; the arc or angle being measured round the equator or the pole from west through south to east, or in the direction of the earth's motion round the sun: thus, in fig. *b*, if X be a heavenly body, and PXR a circle of declination passing through its place, XR is its declination, and AR, or the angle APR, its right ascension. In like manner, if a circle of latitude be drawn through any point in the celestial concave, the part of it between the point and the ecliptic is called the *latitude* of the point, and the arc of the ecliptic extending eastward from the first point of Aries to the circle of latitude is called the *longitude* of the point; thus the latitude of X is XM, and longitude AM. The altitude of a heavenly body is the arc of a circle of altitude intercepted between the place of the body and the horizon; thus XO is the altitude of X. The *azimuth* of a heavenly body is the arc of the horizon intercepted between the north or south points and the circle of altitude passing through the place of the body; or it is the corresponding angle at the zenith between the celestial meridian and the circle of altitude passing through the body: thus SO or NO, or the angles OZS or OZN, is the azimuth of X. The *amplitude* of a heavenly body is the distance from the east at which it rises, or the distance from the west at which it sets, these arcs or distances being measured on the horizon; thus the amplitude of X is the arc WD or ED_2 (the dotted line D_2XD being the arc of a *parallel* of declination described by X from rising at D_2 to setting at D). The *hour angle* of a heavenly body is the angle at the pole between the celestial meridian and the circle of declination passing through the place of the body; thus ZPX is the hour angle of X. If L be the place of the sun *west* of the meridian, its hour angle ZPL is called *apparent* time; but when the sun is *east* of the meridian, as at C, then apparent time is found by subtracting the hour angle ZPC from 24 hours.

---

* If a line be drawn from C, the centre of the earth, to A, and produced to cut the celestial concave in Z_1, the arc Z_1Q is called the latitude *on the sphere*, or the *reduced* latitude.

Definitions in Navigation.

(49). Let A and F represent two places on the surface of the earth (considered as a sphere), PU, PZ, their meridians, P the pole, and UZ an arc of the equator. Through A and F draw a curve line, AF, cutting all the intermediate meridians, PV, PW, &c. *at the same angle.* This common angle is called the *course* from A to F, and the arc AF (in nautical miles) is called the *distance.* Draw the parallels of latitude AA′ and FF′; the arc AU is the latitude of A, and FZ the latitude of F; UZ, or the angle APF, is the difference of longitude between A and F. The arc AF′ is the difference, or as it is called in Navigation, the *true difference of latitude* between A and F.



(50). Suppose the intermediate meridians, PV, PW, &c. to be drawn through points B, C, &c. taken on the arc AF *indefinitely near to one another,* and though A, B, &c. suppose arcs of parallels AM, BN, &c. to be drawn; on this supposition the elementary triangles ABM, BCN, &c. may be considered as *right-angled plane triangles,* of which the sum of the sides AB, BC, &c. is the distance, the sum of the sides BM, NC, &c. is equal to the true difference of latitude, and the sum of the sides AM, BN, &c. is called in Navigation the *departure.*

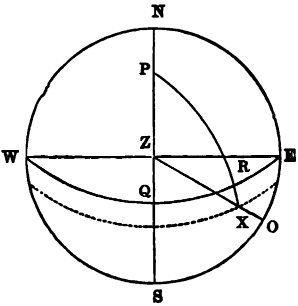
The chart used at sea for marking down the ship's track and for other purposes, exhibits the surface of the globe on a plane, on which the meridians are drawn parallel to each other, and therefore the parts AM, BN, CO, &c., arcs of parallels of latitude, are increased and become equal to the corresponding parts of the equator UV, VW, &c. Now, in order that every point on this plane may occupy the same relative position with respect to each other that the points corresponding to them do on the surface of the globe, the distance between any points A and F′, and A and F, must be increased in the same proportion as the distance FF′ has been increased. The true difference of latitude AF′, is thus projected on the chart into what is called the *meridional difference* of latitude, and the departure AM+BN+CO+ . . into the difference of longitude. A chart constructed in this manner is called a Mercator's Chart. From these definitions and principles are deduced the fundamental formulæ in Navigation; and these, expressed in words, form the common rules of Mercator and parallel sailing. The investigation of these formulæ and rules is given in *Navigation*, Part II.

Spherical Trigonometry derived its origin from the computations necessary in astronomy, and its principal applications are still furnished by the same science.

Examples in Nautical Astronomy.

1. In latitude 30° 45′ N. the altitude of the sun was 14° 45′ (east of meridian), and by the Nautical Almanac its declination at the time of observation was 16° 10′ S; required the azimuth and hour angle.

Let NWSE represent the horizon, and Z the zenith of the spectator; NZS the celestial meridian: then WZE, drawn at right angles to the meridian, will represent the prime vertical (see definitions, p. 124). Take NP an arc of the meridian=30° 45′ the latitude; then P is the pole of the heavens, since the altitude of the pole is equal to the latitude (p. 126): therefore PZ is the colatitude. Take PQ=90°; then Q is a point in the celestial equator. Through W and E,



the west and east points of the horizon, and Q, draw the circle WQE; this will represent the celestial equator.

Let X be the place of the sun at the time of observation. Through X draw ZO a circle of altitude, and PX a circle of declination;* then PZX is a spherical triangle, in which are given the polar distance PX=90° + declination=106° 10′, the zenith distance ZX=90°—altitude=75° 15′, and the co-latitude PZ=90°—latitude=59° 15′; to find the angle PZX the azimuth, and the angle ZPX the hour angle of the heavenly body.

Calculation. (By Rule VIII.)

(1.) To find azimuth PZX, or arc NO.

75°	15′	0·014553
59	15	0·065801
16	0	4·942169
106	10	4·850115
122	10	9·872638
90	10	∴ PZX=119° 27′

or azimuth=N. 119° 27′E.

(2.) To find hour angle ZPX. (By Rule VIII.)

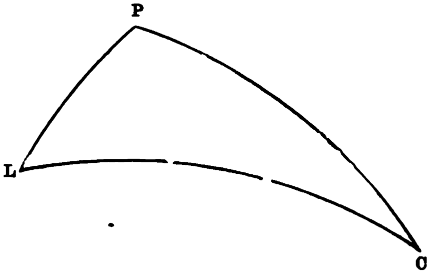
106°	10	0·017523
59	15	0·065801
46	55	4·942169
75	15	4·388711
122	10	9·414204
28	20	∴ ZPX=4h. 5m. 1s. the hour angle.

---

* The place of the heavenly body X in this, and in all the diagrams in the following pages, should be estimated as near the truth as possible *by the eye* without using mathematical instruments; a tolerably correct figure may be always drawn in this manner after a little practice. For instance, since the above figure is projected for a place in north latitude, X must be placed *below* the equator because its declination is south; the altitude XO in the figure is about 15°, and declination RX about 16°, supposing the lines ZO and PR to represent 90°.

2. Required the shortest distance between London and Calcutta: the latitude of London (St. James's Church, Piccadilly) being 51° 31′ N., longitude 0° 8′ W.; the latitude of Calcutta 22° 34′ N., and longitude 88° 26′ E. (the earth considered as a sphere, and the length of a degree=69·05 miles).

Let P represent the north pole of the earth, L London, and C Calcutta. Draw PL and PC, meridians passing through London and Calcutta; and through L and C draw LC, an arc of a great circle: LC is the shortest distance between the two places.



In the spherical triangle LPC are given PL the colatitude of London=38° 29′, PC the colatitude of Calcutta=67° 26′, and the included angle LPC=the difference of longitude=88° 34′; to calculate the third side LC.

Calculation. (By Rule XI.)

To find distance LC.

PC ...	67° 26′	6·301030
PL ...	38 29	9·965406
	28 57	9·793991
P	88 34	9·687968
		5·748395

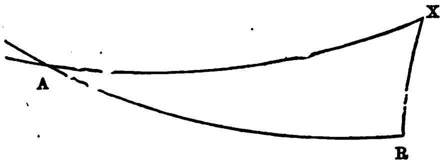
560266
124957
685223

∴ LC=71° 39′=71°·65

∴ distance=71°·65 × 69·05=4947 miles.

3. Required the right ascension and declination of the sun when its longitude is 64° 33′ 15″, the obliquity of the ecliptic being 23° 27′ 45″.

Let AR represent the celestial equator, and AX the ecliptic; the intersection A the first point of Aries. Let X be the place of the sun in the ecliptic: then AX = longitude = 64° 33′ 15″, and the angle XAR = the obliquity of the ecliptic = 23° 27′ 45″. Through X draw XR, a circle of declination: then XAR is a right-angled spherical triangle, in which are given AX, the angle A, and the right angle ARX; to find XR the sun's declination, and AR its right ascension (see definitions, p. 126).



Calculation. (By Rule XIII.)

(1.) To find right ascension AR.

$$\overset{+}{\cos}.\ A = \overset{+}{\cot}.\ AX.\ \overset{+}{\tan}.\ AR$$

AX ...	64° 33′ 15″	19·962521
A	23 27 45	9·677439
		10·285082

∴ AR = 4h. 10m. 20s. the sun's right ascension.

(2.) To find decl. XR.

sin. XR = sin. A sin. AX

9·600045

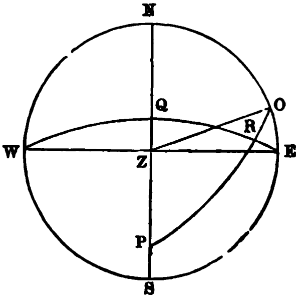
9·955684

9·555729

∴ XR = 21° 4′ 15″ N the sun's declination.

4. Required the time of sun-rising at a place in latitude 30° 45′ S., the sun's declination being 18° 28′ N. (neglecting the effects of refraction, &c.).

Let NWSE represent the horizon, NZS the celestial meridian. Then, since the spectator, whose zenith is Z, is in south latitude, the south pole is above the horizon: take, therefore, SP = 30° 45′ the latitude; then P is the south pole. Let O be the place of the sun in the horizon, and through O draw the circle of altitude ZO and circle of declination PO: then ZPO is a quadrantal triangle, in which are given PZ the colatitude = 59° 15′, the polar distance PO = 90° + declination = 108° 28′, and the quadrant ZO = 90°; to find the angle ZPO, the hour angle at the time of rising, which subtracted from 12h. will give the A.M. time of sun-rising.



To find hour angle ZPO. $\overset{+}{\underline{\text{cos. P}}}=-\overset{+}{\text{cot. PZ}}\ \overset{-}{\text{cot. PO}}$

PZ ... 59° 15′ 9·774471 (By Rule XIV.)

PO ... 100 28 9·523679

∴ P = 5h. 14m. 10s. 9·298150

∴ sun rises at 6h. 45m. 50s. A.M.

The following Problems are given as Exercises in Spherical Trigonometry. They also form an Introduction to Nautical Astronomy.

[102]. In latitude 50° 48′ N. the altitude of the sun was 46° 20′ (west of meridian), and, by the Nautical Almanac, its declination at the time of observation was 23° 27′ 45″ N.; required the azimuth. (Fig. 73.) *Ans.* N. 111° 51′ W.

[103]. In latitude 50° 48′ N. the altitude of the sun was 46° 20′ (west of meridian), and its declination 23° 27′ 45″ N.; required the apparent time of observation. (Fig. 73.)
Ans. 2h. 57m. 16s.

[104]. The azimuth of a heavenly body was observed to be N. 111° 51′ W., its altitude at the same time was 46° 20′, and declination 23° 27′ 45″ N.; required the apparent time. (Fig. 73.) *Ans.* 2h. 57m. 16s.

[105]. What is the altitude of a star whose hour angle is 38° and declination 16° N. at a place in latitude 50° 48′ N.? (Fig. 73.) *Ans.* 43° 49′.

[106]. In latitude 50° 48′ N., when the sun's declination was 12° 29′ N. and hour angle 2h. 53m. 1s. A.M.; required the azimuth. (Fig. p. 129.) *Ans.* N. 121° 47′ E.

[107]. Given the sun's meridian altitude=70° (zenith north of the sun) and its declination 20° N.; required the latitude of the place. *Ans.* 40° N.

[108]. Given the sun's meridian altitude=70° (zenith north) and its declination 5° S.; required the latitude. (Fig. 66.) *Ans.* 15° N.

[109]. Given a star's meridian altitude=70° (zenith south) and declination 25° N.; required the latitude. (Fig. 67.) *Ans.* 5° N.

[110]. Given the sun's meridian altitude 30° (zenith south) and declination 10° N.; required the latitude. (Fig. 68.) *Ans.* 50° S.

[111]. Given the latitude of the spectator 40° N. and the declination of a heavenly body 20° N.; required its meridian altitude. *Ans.* 70°.

[112]. Given the sun's meridian altitude 30° (zenith south of the sun) at a place in latitude 50° S.; find its declination. (Fig. 68). *Ans.* 10° N.

[113]. The meridian altitude of a star at a place on the equator is 57°; find its declination (zenith north of star). *Ans.* 33° S.

[114]. The latitude of a spectator is equal to half the sum of the meridian altitudes of a circumpolar star above and below the pole; required a proof. Given the altitudes 20° and 70° at its inferior and superior transits respectively; required the latitude. (Fig. 69.) *Ans.* 45° N.*

---

* The latitude and declination in the astronomical problems are to be considered *north*, unless the contrary is expressed.

[115]. A circumpolar star passes the zenith of a place, and its altitude at the inferior transit is 20°; required the latitude. (Fig. 70.) *Ans.* 55° N.

[116]. If the altitude of a circumpolar star at its inferior transit is equal to its zenith distance at its superior transit, required the latitude. *Ans.* 45° N.

[117]. In latitude 60° N. what are the altitudes of a star at the inferior and superior transits, its declination being 40° N? *Ans.* 70° and 10°.

[118]. What is the declination of a star that passes the zenith of a place in latitude 50° 48′ N., and what will be its altitude at the inferior transit?

Ans. Decl. 50° 48′ N.; alt. 11° 36′.

[119]. What is the latitude of a place in which the sun's centre just touches the horizon without setting, when its declination is 23° 28′ N.? *Ans.* 66° 32′ N.

(51). The following expressions will be found useful in solving astronomical problems for certain positions of the heavenly body. They are easily deduced from Napier's Rules for right-angled and quadrantal spherical triangles (pp. 72, 75).

1. When the heavenly body is on the *prime vertical*, as at Y, p. 125, then:

α. cos. hour angle = cot. lat. × tan. declination.

β. sin. decl. = sin. lat. × sin. altitude.

γ. cos. alt. = sin. hour angle × cos. decl.

δ. cos. lat. = cot. hour angle × cot. alt.

2. When the heavenly body is *six hours*, or 90°, from the meridian, as at V, p. 125, then:

ϵ. sin. alt. = sin. decl. × sin. lat.

ζ. cos. azimuth = cot. lat. × tan. alt.

η. cos. lat. = cot. decl. × cot. azimuth.

θ. cos. decl. = sin. azimuth × cos. alt.

3. When the heavenly body is in the *horizon*, as at D :

ι. cos. hour angle = —tan. lat. × tan. decl.
κ. sin. decl. = cos. lat. × sin. amplitude.
λ. sin. lat. = —cot. amplitude × cot. hour angle.
μ. cos. amplitude = cos. decl. × sin. hour angle.

[120]. Given the sun's altitude 30° when due west, and its declination 20° N.; required the latitude.

Ans. 43° 9′ 15″ N.

[121]. Given the sun's declination 23° 27′ 45″, and latitude of the place 50° 48′ N.; to find his altitude, and the time when he is on the prime vertical.

Ans. Alt. 30° 55′; time, 4h. 37m. 4s.

[122]. Two stars are due east at the same time at a place whose latitude is 50° 48′ N., and their altitudes are 20° and 40°; find the difference of their hour angles.

Ans. 59m. 56s.

[123]. Given the sun's altitude 18° 45′ at six o'clock, and declination 20° 4′ N.; to find the latitude.

Ans. 69° 31′ 40″ N.

[124]. Given the latitude 50° 48′ N., and sun's declination 23° 27′ 45″ N.; to find his altitude and azimuth at six o'clock. *Ans.* Alt. 17° 58′ 15″; azimuth, N. 74° 39′ 30″ E.

[125]. What will be the apparent time of rising of the sun at a place in latitude 50° 48′ N., when its amplitude is E. 10° S. (neglecting the effects of refraction, &c.)?

Ans. 6h. 31m. 7s.

[126]. Given the sun's amplitude W. 37° 30′ N. and decl. 15° 12′ N.; required the latitude. *Ans.* 64° 29′ 15″ N.

[127]. Given the latitude of the place 50° 48′ N.; and the sun's decl. 18° 28′ N.; required his amplitude, time of rising or setting, and the length of the day and night.

Ans. Amp. E. 30° 4′ 30″ N.; sun rises at 4h. 23m. 19s. A.M.; length of day 15h. 13m. 22s.

[128]. Where will the sun rise in latitude 50° 48′ N. when the day is 14 hours long? *Ans.* E. 19° 5′ N.

[129]. Given the sun's altitude 22° 56′, and hour angle 3h. 0m., when the declination is nothing, to find the latitude.
Ans. 56° 33′ 30″ N.

[130]. Given the sun's altitude 42° 30′; declination, 22° 10′ N.; and azimuth, S. 57° 45″ W.; to find the latitude. (Fig. 74.) *Ans.* Lat 59° 4′ N.

[131]. Given the sun's altitude 37° 20′; hour angle 2h. 15m., and decl. 22° 30′ N.; to find the latitude (zenith north of the sun). (Fig. 74.) *Ans.* Lat. 71° 31′ N.

[132]. Given the right ascension of the sun (RA), 4h. 10m. 20s.; and obliquity of the ecliptic (ω), 23° 27′ 45″; to find his longitude and declination. (Fig. 76.)
Ans. Long. 64° 33′ 15″; decl. 21° 4′ 15″ N.

2. Let RA=17h. 10m.; ω=23° 27′ 45″. (Fig. 77.)
Ans. Long. 258° 30′ 15″; decl. 22° 58′ S.

[133]. Given the right ascension and declination of a heavenly body, and the obliquity of the ecliptic; to find its latitude and longitude. (Figs. 77, 78.)

EXAMPLES.

1. Let RA=2h. 59m. 37s.; decl.=21° 27′ 48″ N.; ω= 23° 27′ 45″. *Ans.* Lat. 4° 15′ N.; long. 48° 37′ 30″.

2. Let RA=16h. 14m.; decl.=25° 51′ N.; ω=23° 27′ 45″. *Ans.* Lat. 46° 6′ 15″ N.; long. 234° 36′ 30″.

[134]. Two places have the same latitude, namely, 45° N., and their difference of longitude is 10° 36′; required their distance. (Fig. 79.) *Ans.* 449·25 nautical miles.

[135]. Required the distance from Portsmouth to Buenos Ayres. (Fig. 80.)

Lat. Portsmouth, 50° 48′ N.; long. 1° 6′ W.
„ Buenos Ayres, 34 37 S.; long. 58 24 W.

Ans. 5949·8 nautical miles, or 6847·2 English miles ($69\frac{1}{20}$ to a degree).

Sailing on or near to a Great Circle.

(52). The shortest distance between two known places is the arc of a great circle passing through them. This distance may be found as in the last example. The practical inconvenience of sailing on a great circle arises from the necessity of continually altering the course. It is for this reason that Rules are given for sailing from one place to another so that the course may be constant, although the distance sailed is not the shortest between the two places. When the distance between the two places is considerable, as between Liverpool and New York, the following method of approximating to the shortest distance may be adopted with advantage:

1. Compute the shortest distance as in the last example.
2. Take two or more points on this arc, and find the latitude and longitude of those points.*
3. Find the course and distance from the place sailed from, to the nearest point marked on the arc, and thence to the next point, and so on. (*Navigation*, Part I.)

The distance described by the ship on these curves will not differ much from the shortest distance; and thus the advantage of sailing on one course is combined with that of shortening the distance. This will be seen in the following example:

Example in Great-circle Sailing.

1. Compute the shortest distance between Liverpool and New York.†

Lat. Liverpool, 53° 25′ N.; long 2° 59′ W.
„ New York, 40 42 N.; „ 73 59 W.

Ans. 2872 miles.

---

* The points should not be farther apart than 1000 miles. The nearer they are taken to each other, the less will be the difference between the sum of the distances run and the shortest distance.

† This example is worked out at length in *Navigation*, Part II. pp. 167-169.

2. Take a point A, on this arc, distant from Liverpool 872 miles; take another point B, distant from A=1000 miles; then from B to New York=1000 miles. Find the latitude and longitude of the points A and B.

Ans. Lat. A, 54° 39′ N.; long. 27° 46′ W.
„ B, 50 0 N.; „ 54 9 W.

3. Find course and distance from Liverpool to A, from A to B, and from B to New York (see *Nav.* Part I.).

Ans. Course and distance from :

Liverpool to A	N. 85° 9′ W.;	dist.	876 miles.
A to B.	S. 73 54 W.;	„	1006 „
B to New York ...	S. 56 11 W.;	„	1002 „

4. Compute the difference between the distance, by great-circle sailing, from Liverpool to New York, and the distance by Mercator's sailing. *Ans.* 107 miles.

---

[136]. Required the distance of the moon from α Leonis (Regulus); the right ascension and declination of the former being 0° 32′ 45″ and 5° 19′ S., and of the latter 148° 18′ 45″ and 13° 10′ 15″ N. (Fig. 81.) *Ans.* 147° 16′.

[137]. The meridian altitude of α Leonis (declination 13° 10′ N.) under the North Pole was observed to be 6° 10′; required the latitude. *Ans.* 83° N.

[138]. Given the sun's longitude 202° 24′ 15″, and the moon's latitude and longitude 4° 54′ 30″ N. and 89° 25′ 30″; required the distance. (Fig. 84.) *Ans.* 112° 53′ 30″.

[139]. If a ship from latitude 50° 10′ N. start on a S.W. course, and sail 100 miles on a great circle, what will be her last course? (Fig. 85.) *Ans.* S. 43° 38′ W.

[140]. The latitude of a place A is 40° N., of B 50° N., and their distance from each other 20°, the longitude of A is 15° E.; required the latitude and longitude of another place

C to the north of and 20° distant from A and B. (Figs. 86, 87.) *Ans.* Lat. 59° 37′ N.; long. 21° 13′ E., or 8° 47′ E.

[141]. The distances of a comet from Aldebaran and Regulus were observed to be 40° 12′ and 51° 36′ respectively; required its latitude and longitude, the respective latitudes of the two stars being 5° 28′ 45″ S. and 0° 27′ 30″ N., their longitudes 67° 12′ 15″ and 147° 15′ 30″, and the comet being south-east of the arc of a great circle joining them. (Fig. 88.) *Ans.* Lat. 28° 0′ 15″ S.; long. 102° 19′.

[142]. Required the beginning of morning and the end of evening twilight at a place in latitude 54° 36′ N., the sun's declination being 8° 30′ N. (twilight being supposed to begin and end when the sun is 18° below the horizon). (Fig. 83.)
Ans. 2h. 46m. A.M., 9 h. 14m. P.M.

[143]. Suppose two altitudes of the sun observed in the forenoon in the same place, at the interval of an hour and a half, to be 28° 40′ and 39° 50′; required the latitude of the place, the declination being 23° 26′ N. at both observations. (Fig. 89.) *Ans.* Lat. 59° 17′ 15″ N.

[144]. The altitudes of α Hydræ and Regulus (both east of meridian) were observed at the same time to be 40° 44′ and 45° respectively, the right ascension and declination of the former being 9h. 16m. and 7° 37′ S., of the latter 9h. 53m. and 13° 9′ N.; required the latitude. (Fig. 91.)
Ans. Lat. 26° 23′ N.

[145]. The difference of longitude between two places in the same latitude, namely 33° 51′ S., is 136° 10′; how much shorter is the distance between them on the arc of a great circle than on their common parallel, the distance measured on the parallel being 6785 miles; and what is the highest latitude attained by a ship sailing from one to the other on the arc of a great circle? (Fig. 92.)

Ans. Difference of distances, 737·6 nautical miles; highest lat. 60° 54′ S.

[146]. What is the highest latitude attained by a ship sailing on the arc of a great circle from Port Jackson to Cape Horn, their latitudes being 33° 51′ and 55° 58′, and the difference of longitude 140° 27′? (Fig. 93.) *Ans.* 72° 41′.

[147]. Required the sun's azimuth and depression below the horizon at 7h. P.M., when the declination was 10° 15′ S., and the latitude of the place 50° 48′ N. (Fig. 83.)

Ans. Azimuth N. 84° 53′ W.; depression, 17° 24′.

[148]. Determine the bearing or azimuth of the two stars Aldebaran and Pollux when on the same vertical circle, the latitude of place being 25° N.; the R.A. and declination of former star being 4h. 26m. 46s. and 16° 11′ N.; of the latter 7h. 35m. 13s. and 28° 24′ 26″ N. (Fig. 94.)

Ans. Bearing of Aldebaran, S. 75° 5′ W.

[149]. In a certain latitude (zenith N.) the moon's true altitude was 18° 2′ 30″ (east of meridian) in the same vertical circle with a star whose right ascension and declination were 9h. 59m. 42s. and 12° 45′ 45″ N., the moon's R.A. and declination being 12h. 35m. 54s. and 1° 42′ 30″ S.; required the latitude. (Fig. 95.) *Ans.* 19° 55′ 30″ N.

The following problems are given as exercises in *analytical* trigonometry.

[150*a*]. The altitude of a star when due east was 20°, and it rose E.b.N.; required the latitude. *Ans.* 29° 42′ N.

[151*a*]. The altitude of a star when due east was 10°, when due south was 40°; find the latitude of the place.

Ans. 58° 31′ N.

[152*a*]. Given the altitude of the sun when due west and at six o'clock, to find the latitude and declination.

EXAMPLE.

Altitude when west=27° 24′; at 6h.=14° 43′ 30″.

Ans. Latitude, 48° N.; declination, 20° N.

[153*a*]. Given the sun's altitude at six o'clock, and its amplitude; to find the latitude and declination.

EXAMPLE.

Altitude at 6h.=14° 43′ 30″; amplitude=30° 44′ 30″.
Ans. Latitude, 42° or 48°; declination, 22° 20′ or 20°.

[154*a*]. Given the sun's altitude at six o'clock, and the hour angle when setting; to find the latitude and declination.

EXAMPLE.

Altitude at 6h.=14° 43′ 30″; h=7h. 35m. 22s.
Ans. Latitude, 48° 1′; declination, 20°.

[155*a*]. Given the times at which the sun sets and is west on the same day, at a particular place; to find the latitude of the place, and the declination.

EXAMPLE.

Hour angle when W.=4h. 43m. 28s.
setting=7h. 35m. 22s.
Ans. Latitude, 48° N.; declination, 20° N.

[156*a*]. Given the sun's declination, and the interval between the times at which he is west and sets; to find the latitude.

EXAMPLE.

Declination=20° N.; interval=2h. 51m. 54s.
Ans. Latitude, 48° N. or 42° N.

[157*a*]. Given the amplitude of the sun, and the azimuth at six o'clock; to find the latitude and declination.

EXAMPLE.

Ampl.=W. 30° 44′ 30″ N.; azimuth=N. 76° 18′ 45″ W.
Ans. Latitude, 48° N.; declination, 20° N. nearly.

[158*a*]. Given the sun's meridian altitude, and his altitude at six o'clock; to find the latitude and declination.

EXAMPLE.

Mer. alt.=62°; alt. at 6h.=14° 43′ 30″
Ans. Latitude, 48° 0′ 20″ N.; declination, 20° N.

[159*a*]. Given the sun's meridian altitude, and the hour angle when rising; to find the latitude and declination.

EXAMPLE.

Mer. alt. = 56°; hour angle = 7h.

Ans. Latitude, 47° 23′ 30″.

[160*a*]. Given the interval between the times at which the sun is west and sets at a place whose latitude is known; to find the declination.

EXAMPLE.

Lat. = 48° N.; interval = 2h. 51m. 54s.

Ans. Declination, 20° N.

[161*a*]. At a given place, to find the greatest azimuth of a heavenly body, whose declination is greater than the latitude of the place; to find also the time and altitude on a given day, when the heavenly body will have the greatest azimuth, and when consequently it will appear to move perpendicularly to the horizon.

EXAMPLE.

In lat. 20° N. when the sun's declination is 23° 28′ N.; required the time and altitude when its azimuth is the greatest, and also its greatest azimuth. (Fig. 96.)

Ans. Time, 9h. 47m. 53s. A.M. (app. time); altitude, 59° 11′ 30″; greatest azimuth, N. 77° 28′ E.

[162*a*]. In latitude 20° N. when the sun's declination is 23° 28′ N.; required the time when the sun will appear stationary in azimuth, the period during which the shadow moved in a contrary direction, and the number of degrees it appeared to go back.

Ans. Time, 9h. 47m. 53s.; period, 4h. 24m. 15s.; shadow went back 12° 32′ 30″.

[163*a*]. When the sun's declination was 10° 15′ N., and that of the moon was 12° 46′ S., both were observed to rise at the same time; required the latitude of the place and the

time of the observation, the difference of their R.A.'s being 1h. 53m. 42s. (Fig. 97.)

Ans. App. time, 5h. 9m. 40s. A.M.; lat. 50° 18′ 20″ N.

[164*a*]. On what days of the year is the sun on the horizons of Dublin and Pernambuco at the same instant, their respective latitudes being 53° 21′ N. and 8° 13′ S., and their longitudes 6° 19′ W. and 35° 5′ W.?

Ans. The four days in the year when the sun's declination is 18° 6′.

[165*a*]. ABCD is a square field, each of whose sides is 100 yards. In the middle of the field stands an obelisk 60 feet high; find the altitude of the sun when the shadow of the obelisk just reaches the corner of the square. *Ans.* 15° 47′.

[166*a*]. At noon on the shortest day the shadow of a perpendicular stick was 7 times as long as its shadow at noon on the longest day; required the latitude, the declination being 23° 28′. (Fig. 98.) *Ans.* Sin. $2l=\frac{4}{3}$ sin. $2d$; $l=38°\ 27'\ 45''$.

[167*a*]. Compare the lengths of the shadow of a perpendicular stick at noon in latitude 45° N. on the two days when the sun's declination was 15° N. and 15° S.

Ans. Length of shadows as 3 : 1.

[168*a*]. In latitude 33° 30′ N. and decl. 10° 15′ N., I observed that my shadow bore to my height the proportion of 5 : 3; required the altitude and hour angle of the sun.

Ans. Alt. 30° 58′; h=3h. 58m. 4s.

[169*a*]. The length of the shadow of a perpendicular object was 4 feet, and its longest when sloping was 5 feet; required the sun's altitude. (Fig. 99.) *Ans.* 36° 52′ 15″.

[170*a*]. The elevation of a cloud was observed to be 20°, at the same time the sun's altitude was 22°, the sun and cloud being in the same plane with the observer, and his distance from the shadow 400 yards; required the height of the cloud. (Fig. 100.) *Ans.* 1468 yards.

[171*a*]. In latitude 45° N, the meridian altitude of the

sun$=30°$; show that the tangent of quarter the length of the day$=\frac{1}{\sqrt{3}}$. (Fig. 101.)

[172*a*]. At a certain place the sun rose at 7h. A.M., and its meridian zenith distance was twice the latitude; required the latitude. *Ans.* Lat. 26° 58′.

[173*a*]. In latitude 45° N. the sun rose at 4h. A.M.; show that the tangent of the meridian altitude$=3$.

[174*a*]. In latitude 50° N. when the sun's declination is 5° 38′ N., required the time it will take the body of the sun to rise out of the horizon, its semidiameter being 16′. (Fig. 102.) *Ans.* 3m. 19s.

[175*a*]. Required the time the sun's semidiameter will take to pass the meridian, the declination being 23° 4′ and semidiameter 16′ 17″·3. (Fig. 103.) *Ans.* 1m. 11s.

[176*a*]. In latitude 59° 6′ N. and long. 6° 15′ E. observed a point of land bearing N.E., and after sailing E.N.E. 6 miles the point bore N.$\frac{1}{2}$W.; required the latitude and longitude of the point. (Fig. 104.)

Ans. Lat. 59° 11′ 15″; long. 6° 25′ 13″ E.

[177*a*]. In latitude 45° N., required the difference in the lengths of the longest and shortest days (declination 23° 28′).

Ans. 6h. 51m. 40s.

[178*a*]. In what latitude N. will the difference between the longest and shortest days be just 6 hours?

Ans. Lat. 41° 24′ N.

[179*a*]. At a certain place, when the sun's declination was 10° N., it rose an hour later than when it was 20° N.; required the latitude. *Ans.* Lat. 52° 27′ N.

[180*a*]. In what latitude N. will the shortest day be just one-third the longest (declination 23° 29′)?

Ans. 58° 27′ N.

[181*a*]. And in what latitude will the shortest day be just $\frac{3}{5}$ths the longest? *Ans.* 41° 24′ N.

[182*a*]. If a ship sail from a certain place *a* miles due

east, then a miles due south, and then b miles due west, and reach the same longitude; required the latitude of the place arrived at.

EXAMPLE.

$a=100'$, and $b=150'$. (Fig. 130.) *Ans.* 85° 0′ 30″.

To find the true bearing of a Terrestrial object.

[183*a*]. Given the apparent and true altitude of a heavenly body, its declination, and observed distance from a terrestrial object; to find the true bearing of the object from a given station. (Fig. 106.)

1. When the object is in the horizon.

In latitude 7° 51′ N. the observed altitude of the sun's lower limb was 10° 30′ west of meridian, and observed distance of his nearest limb from a well-defined point of land on the same level with the eye and to the right of the sun was 95° 16′; index correction of the altitude sextant was 50″—, and that of the other was 1′ 10″+; the correction for height of the eye (14 feet) in taking the sun's altitude was 3′ 41″—; required the true bearing of the point of land, the sun's declination being 22° 24′ S. and semidiameter 15′ 45″.

Ans. Bearing, N. 19° 0′ 30″ W.

2. When the object is elevated above the level of the eye it is necessary to observe its altitude.

In the preceding example, suppose the object observed to be a mountain, the altitude of whose summit is 10°; required the true bearing of that point. *Ans.* N. 17° W.

Measurement of an Arc of the Meridian.

(53). The system of rules and operations by which the relative position of any number of points in a tract of country may be determined, so that it may be delineated on a plane surface, is called *Trigonometrical Surveying.* When the extent of country is not great, the subject involves little difficulty; but when a kingdom such as Britain or France is to be surveyed, the aid of astronomy and other branches of natural philosophy is required. In a survey, the most remarkable objects, such as the summit of hills, spires, towers, &c. must be chosen as *stations*, and, if necessary, marked by signals. These must be considered as joined by straight lines, forming a chain of triangles, connecting each with all the others. The sides of the triangles should be as long as possible, so as to admit of the stations at any two of the angles being seen from the third; the nearer each triangle approaches to the equilateral form the better. Supposing a proper disposition of the triangles to have been made, when their angles are known, if a side of any one of them were also known, then the sides of all the others might be found by calculation, and a plan of the country constructed.

A small extent of the earth's surface may be regarded as a plane, and lines perpendicular to it as parallel to one another. In an extensive survey, however, such as that of England, the curvature of the earth must be taken into the account, and then its *figure* and *magnitude* enter as elements into all the calculations. This connection between the figure of the earth and the magnitude and position of lines traced on its surface affords conversely the means of determining the former when the latter are known; so that such surveys, besides their immediate object, are applicable to the solution of the still more important problem of finding the magnitude and figure of the earth itself. (See p. 151.)

When a tract to be surveyed has been covered with a series of triangles, so as to connect the principal points, and all the angles of each and a side of one are known, the sides of all the triangles may be found by calculation, and a plan made by constructing the triangles on the sides of each other; but in a plan so constructed any error made in the value of one of the sides or angles will produce corresponding errors in all the others.

To avoid as much as possible this source of error, we may determine the position of a side of one of the triangles with respect to the meridian by compass, or more accurately by astronomical observations (see Problem 183); and then calculate the distances of all the stations from such meridian, and also the meridional distance of one station from the other. From these distances the position of each station may be laid down in the plan independently of the others, and also the direct distance between any two points may be easily found. The manner of proceeding will appear from the following Problem.

[184]. (Fig. 59.) Let A, B, C, D, E, F, be six stations connected by four triangles ABC, BCD, BDE, EDF; the angles are:

BAC=79°20′
ABC=51 31
ACB=49 9
DBE=45 28
BDE=72 3
BED=62 29

CBD=39°20′
BCD=69 28
BDC=71 12
EDF=62 3
DEF=52 25
DFE=65 32

A side, AB, of one of the triangles is 4213 yards, and it makes with the meridian, NS, an angle SAB=62° 52′ at the point A; and the station F makes at the point S an angle ASF with the meridian=52° 40′. It is required to find the points in which perpendiculars from the stations will cut the meridian, and the length of each perpendicular.

SOLUTION.

1. Draw B*b*, C*c*, D*d*, E*e*, F*f*, perpendicular to the meridian, and B*n*, D*q*, parallel to it, forming the right-angled triangles AB*b*, BC*m*, BD*n*, ED*p*, FD*q*. Because the angles of the four triangles ABC, BCD, BDE, DEF, are given, and also AB, a side of one of them, the five lines AB, BC, BD, DE, DF, may be found from AB and each other (their logarithms are:

AB=3·624591, BD=3·733559, DF=3·578533,
BC=3·738255, and DE=3·638690).

2. In the right-angled triangle AB*b*, the side AB (or its

log.) and the angle $BAb=62^\circ\ 52'$ are known, hence we find $Ab=1921{\cdot}4$, $Bb=3749{\cdot}3$ yards.

3. If from $ABm=117^\circ\ 8'$ (the supplement of BAb) the angle $ABC=51^\circ\ 31'$ be taken, there remains $CBm=65^\circ\ 37'$; therefore, in the right-angled triangle CBm the angles and the log. of BC are now known, hence we find $Bm=2259{\cdot}6$, and $Cm=4985{\cdot}2$.

4. And if from $CBm=65^\circ\ 37'$ $CBD=39^\circ\ 20'$ be subtracted, there remains $DBn=26^\circ\ 17'$; therefore in the right-angled triangle DBn the angles and the log. of BD are known, and hence $Bn=4854{\cdot}7$ and $Dn=2397{\cdot}6$.

5. From $BDq=153^\circ\ 43'$ (supplement of DBn) subtract $BDE=72^\circ\ 3'$, and there remains $EDp=81^\circ\ 40'$; then in the right-angled triangle EDp the angles and the log. of DE are known, and thence $Dp=630{\cdot}7$, $Ep=4306{\cdot}1$.

6. Subtracting $EDF=62^\circ\ 3'$ from $EDp=81^\circ\ 40'$, there remains $FDq=19^\circ\ 37'$; and hence in the right-angled triangle DFq we find $Dq=3569{\cdot}1$, $Fq=1272{\cdot}1$.

To determine the stations we have now :

Ab .... $= 1921{\cdot}4$	Bb .... $=3749{\cdot}3$
$Ac=Ab+Bm= 4181{\cdot}0$	$Cc=Cm-Bb=1235{\cdot}9$
$Ad=Ab+Bn= 6776{\cdot}1$	$Dd=Bb-Dn=1351{\cdot}7$
$Ae=Ad+Dp= 7406{\cdot}8$	$Ee=Dd+Ep=5657{\cdot}8$
$Af=Ad+Dq=10345{\cdot}2$	$Ff=Dd+Fq=2623{\cdot}8$

By these numbers, the position of each station may be laid down with great accuracy in a plan, independently of the others; also the distance between any two may be readily found; for example:

$$CE=\sqrt{\{(Cc+Ee)^2+(Ae-Ac)^2\}}$$

Lastly. We know by observation that the bearing of F from S is N. 52° 40′ E.; hence in the right-angled triangle FfS we find $Sf=2001{\cdot}2$ yards; adding this to the line Af, we have the length of the meridian line between the stations A and $S=12346{\cdot}4$ yards.

The above method of measuring the meridian line between two distant stations furnishes, as we have said, the means of finding the approximate length of a degree, and thence, on the supposition of the earth being an exact sphere, the magnitude of the earth; for we have only to determine by observation the difference of latitude between the stations A and S, and then divide the whole length of the arc, found as above, by the number of degrees contained in it; the result multiplied by 360 will give the circumference of the earth. Thus, let us suppose the latitude of A has been found by some of the preceding problems to be 49° 48′ 6·4″ N., and that of S 49° 42′ 4·86″ N., and the length of the line AS to be 12346·4 yards; then by a common proportion we easily find that the length of a degree is about 69 miles. Picard, in 1670, by observations and measurements conducted in a manner similar to the above, found the length of a degree of the meridian in latitude 49° N. to be 121627 yards, which differs only 35 yards from what is now considered as the more exact length; an accuracy, however, which must be supposed to be quite accidental.

If we assume the above measurement to be correct, and the earth to be a *sphere*, its circumference would be about 25,000 miles. Similar observations and measurements have, however, since been made, in the same and other parts of the earth, with more accurate instruments; from which it appears that the lengths of a degree of latitude are not invariable, as would be the case if the earth were a sphere; but that they *increase from the equator to the poles;* suggesting to us the figure of an oblate spheroid.

The several lengths of a degree of the meridian, measured in Sweden, Russia, England, India, &c. are given in the Author's work on Navigation, Part II., and also the investigation of the following formulæ for determining by means of these measurements the lengths of the equatorial and polar diameters of the earth. By applying the formulæ to the observed lengths of a degree in different parts of the earth, we find that the semi-major and semi-minor diameters of the earth are about 3962 miles and 3950 miles respectively.

Formulæ for determining the Earth's Equatorial and Polar Diameters.

Let D and D_1 be the lengths (in feet) of a degree of the meridian in latitudes l and l_1 respectively, a and b the semi-major and semi-minor diameters (expressed in miles), then

$$a-b=\cdot 0036171\,(\mathrm{D}-\mathrm{D}_1)\ \text{cosec.}\,(l+l_1)\ \text{cosec.}\,(l-l_1)$$

$$a=\cdot 0036171\left\{2(\mathrm{D}-\mathrm{D}_1)+3(\mathrm{D}_1\sin.^2 l-\mathrm{D}\sin.^2 l_1)\right\}\text{cosec.}(l+l_1)\text{cosec.}(l-l_1)$$

EXAMPLES.

[184*a*]. The lengths of a degree of the meridian measured in England and India in latitudes 52° 35′ 45″ N. and 12° 32′ 21″ N. were 364971 feet and 363013 feet respectively; required the equatorial and polar semi-diameters a and b.

Ans. $a-b=12{\cdot}2$ miles; $a=3962$ miles, $b=3950$ miles.

[185*a*]. The length of a degree in Russia in lat. 58° 17′ 37″ N. is 365368 feet, and in India, lat. 16° 8′ 22″ N. is 363044 feet: calculate the equatorial semi-diameter of the earth.

Ans. 3961·8 miles.

LONDON
ROBSON AND SONS, PRINTERS, PANCRAS ROAD, N.W.

APPENDIX:

BEING

THE SOLUTIONS OF 180 PROBLEMS

IN

ASTRONOMY, SURVEYING,

AND

NAVIGATION,

CONTAINED IN TRIGONOMETRY, PART I.

By H. W. JEANS, F.R.A.S.

LONDON:
LONGMANS, GREEN, READER, AND DYER.
1878.

PREFACE.

The following solutions of the problems contained in Trigonometry, Part I., are given chiefly with the object of facilitating *self-instruction* in the principles of Navigation and Nautical Astronomy. Designed for the use of all who are interested in navigating ships, they are especially adapted for those who have to rely on their own exertions for acquiring a theoretical knowledge of these subjects.

Langstone, Nov. 1878.

APPENDIX.

PROB. 1 (Fig. 17).

THIS problem is similar to the one solved in p. 83.*

Let C*b* represent the height of the tower=216, A*a* height of observer's eye=5, and *ab* the width of river required.

Draw AB parallel to horizontal line *ab*, $\therefore$ B*b*=A*a*=5, and CB=C*b*−B*b*=216−5=211; join AC; then CAB= angle of elevation=47° 56′. In the right-angled triangle ABC are given the perpendicular CB=211, and angle CAB =47° 56′, to find the base AB=*ab*, the width of river required.

(Rule V. p. 57) $\frac{\text{AB}}{\text{BC}}$=cot. CAB, or AB=BC . cot. CAB= 190·4.

PROB. 2 (Fig. 53).

This problem is similar to the one solved in p. 83.

In ABD, $\frac{\text{BD}}{\text{AD}}$=cot. B, or BD=AD . cot. B=133·7 (p. 57)

In ADC, $\frac{\text{DC}}{\text{AD}}$=cot. C, or DC=AD . cot. C=71·3

and BC=BD+DC=133·7+71·3=205, the *Answer.*

PROB. 3 (Fig. 17).

This problem is similar to the one solved in p. 83.

To find BC (p. 57), $\frac{\text{BC}}{\text{AB}}$=tan. A, or BC=AB . tan. A=471·1

„ AC, $\frac{\text{AC}}{\text{AB}}$=sec. A, or AC=AB . sec. A=511·9

---

* The references are to Trigonometry, Part I., sixth edition.

∴ perimeter of triangle $=BC+AC+AB=1183$ feet $=\cdot224$ of a mile. ∴ $4:\cdot224::60^{m}:$ time required $=3^{m}\ 21\cdot6^{s}$. *Ans.*

Prob. 4 (Fig. 53).

This problem is similar to the one solved in p. 83.

In triangle BAC, AC=AB, and AD, the perpendicular upon BC, bisects the angle BAC and side BC.

DC=50, and angle DAC=75° are given, to find AC.

$\frac{AC}{DC}=\text{cosec. } DAC$, or $AC=DC\ .\ \text{cosec. } DAC=51\cdot77$. *Ans.*

Prob. 5 (Fig. 18).

Let AC represent the broken piece=39, A the point where it struck the ground, and AB the distance of its top from the base=15. Representing the known quantities AC by a, and AB by b, and the unknown part BC left standing by x, we have in right-angled triangle ABC, $x^2=a^2-b^2=(a+b)\,.\,(a-b)=54\times24$ ∴ $x=\sqrt{54\times24}=36$.. (p. 13), and whole height of pole $=39+36=75$ feet. *Ans.*

Prob. 6 (Fig. 19).

Let D and E be the windows, DC=EC the ladder $=36=a$, BD height of lower window $=18\cdot9=b$, and AE height of other window $=30\cdot7=c$. Draw FD parallel to AB, then FDE is the angle of elevation of E above D.

Let $BC=x$, $AC=y$, then $x+y=$ breadth of street.

In ACE, CBD, $y^2=a^2-c^2=(a+c)\,.\,(a-c)$ and $x^2=a^2-b^2=(a+b)\,.\,(a-b)$, whence $y=18\cdot8$, and $x=30\cdot7$, as in last problem.

∴ breadth of street $=x+y=49\cdot5$ feet.

and tan. $FDE=\frac{EF}{FD}=\frac{c-b}{x+y}=\frac{11\cdot8}{49\cdot5}=13^\circ\ 24'\ 30''$. *Ans.*

Prob. 7 (Fig. 17).

Similar to problem p. 83.

Prob. 8 (Fig. 20).

This problem is similar to the one solved in p. 84.

Ex. 1. Let C be the place of the ship, AB the base=1000 yards, A=32° 10′, and B=83° 18′. In triangle ABC are given the side AB or c, and the two angles A and B, whence the third angle C=64° 32′, and the side CA or b, the distance of the ship from A, may be found by Rule p. 47.

$$\text{thus } b : c :: \sin. B : \sin. C \ \therefore\ b = 1100{\cdot}1 \text{ yards.}$$

Ex. 2. Since $A+B=80^\circ+50=130^\circ \therefore C=50^\circ=B$

$$\therefore \text{side } CA=AB=2{\cdot}5 \text{ miles.}$$

Prob. 9 (Fig. 20).

This problem is solved in a similar manner to the last.

To find AC .. $AC : AB :: \sin. B : \sin. C \ \therefore\ AC=9{\cdot}478$ miles.

To find BC .. $BC : AB :: \sin. A : \sin. C \ \therefore\ BC=9{\cdot}52$ „

Prob. 10 (Fig. 53).

Let ABC be an isosceles triangle, AB=AC, and A the vertical angle. Draw AD perpendicular to BC, then AD bisects the vertical angle and also the base BC; therefore in the right-angled triangle ADC are given DC=60, and angle DAC=23° 30′, to find AC.

$$\frac{AC}{DC}=\text{cosec. } DAC, \text{ or } AC=DC\,.\,\text{cosec. } DAC=150{\cdot}5=AB.$$

Prob. 11.

In Fig. 17, draw a line BD from the right angle B perpendicular to AC: it is required to find BD, having given AB=18 and BC=24.

(1.) In right-angled triangle ABC, given BC and AB, to find angle A,

$$\tan. A=\frac{BC}{AB} \therefore A=53^\circ\ 7'\ 45''.$$

(2.) In right-angled triangle ABD, given AB and angle A, to find BD,

$$\frac{BD}{AB}=\sin. A \therefore BD=14{\cdot}4. \ \textit{Ans.}$$

PROB. 12 (part of Fig. 21).

This problem is similar to the one solved in p. 88.

Let AB represent the height of tower, D and C the stations; then DC=100, ACB=46° 15′, and ADB=31° 20′, are given, to find AB.

(1.) In oblique-angled triangle ADC are given DC=100, angle ADB=31° 20′, and angle DAC=(ACB−ADC)=14° 55′, to find AC.

(By p. 47.) AC : DC :: sin. D : sin. DAC ∴ AC=202.

(2.) In right-angled triangle ABC are given AC=202, and ACB=46° 15′, to find AB.

$\frac{AB}{AC}$=sin. ACB, or AB=AC sin. ACB ∴ AB=145·9. *Ans.*

PROB. 13 (see Ex. p. 88).

The solution of this problem is similar to the last.

Let AB (Fig. 21) represent the height of cloud, D and C the two ships; then DC=880 yards, ADC=35°, ACB=64° ∴ DCA=116°, and DAC=29°.

(1.) To find AC, or rather log. AC,

AC : DC :: sin. ADC : sin. DAC ∴ log. AC=3·017503.

(2.) To find AB,

$\frac{AB}{AC}$=sin. ACB, or AB=A C . sin. ACB ∴ AB=935·6,

and height of eye above the sea=935·6+7=942·6 yards. *Ans.*

PROB. 14 (part of Fig. 21).

Let AB represent the tower, C and D the stations; then CB=200, DB=350, and angle ACB=39° are given, to find AB and angle ADB.

(1.) To find AB, $\frac{AB}{CB}$=tan. ACB ∴ AB=162 feet nearly.

(2.) To find ADB, tan. ADB=$\frac{AB}{DB}$, whence ADB=24° 50′.

Prob. 15 (Fig. 20).

Let B and C represent the two ships, A the spectator; then distances AC=4568 and AB=6852 (being 1142 × elapsed seconds); hence in the triangle ABC are given two sides, and included angle A=48° 42′, to find the third side CB. ∴ by Rule p. 54 or 55 the side CB=5147·9 feet.

Prob. 16 (Fig. 21′).

Let AB represent the ship's mast, C the other ship's hull; then, if BD is a horizontal line, DBC is the angle of depression, and is equal to the alternate angle BCA. In the right-angled triangle BAC are given AB=80, angle C=20°; hence AC=219·8 feet.

Prob. 17.

Draw two parallel lines AB, CD, perpendicular to a horizontal line BD, to represent the two monuments; let AB=100, CD=50, and through the tops A and C draw a straight line cutting the horizontal line BD produced in E: then angle E=37°; also since AB=2CD ∴ BE=2DE and DE=BD. To find DE or BD. In the right-angled triangle CDE are given CD=50 and E=37°; hence DE and ∴ BD=66·4 feet.

Prob. 18 (Fig. 23).

Let AB represent the obelisk, C and D the stations.

(1.) In triangle ADC, given DC and all the angles, to find AC.

(2.) In triangle ACB, given BC and AC and included angle ACB, to find AB, the height of obelisk=57·62 feet.

Prob. 19 (Fig. 24).

This useful problem is similar to the one solved in p. 89.

(1.) In triangle CAB are given the angles CAB, ABC, and base AB, to find AC=706·8. (2.) In triangle DAB are given the angles DAB, DBA, and base AB, to find AD$=1389{\cdot}2$. (3.) In triangle ADC are given the two sides AC, AD, and included angle CAD (the difference between CAB and DAB), to find CD$=1174{\cdot}4$ yards.

PROB. 20 (Fig. 24).

This problem is similar to the preceding.

PROB. 21.

Let ACB be the right-angled triangle resting upon its hypothenuse, C the right angle $\therefore c=100$ feet. Let A be the given angle$=36^\circ\ 40'$. Let AC$=x$, and CD, a perpendicular from C upon base AB$=y$; and when the plane of the triangle is inclined at an angle of 60° to the horizon, let the perpendicular distance of point C from the horizontal plane be denoted by z. In this case z will be the perpendicular in a right-angled triangle of which y is the hypothenuse, and angle at base$=60^\circ$. (By turning down the left-hand upper corner of this leaf a right-angled triangle will be formed; the creased line may represent the hypothenuse c, the corner of the leaf the right angle C, and the other two sides x and y: then, if this right-angled triangle be inclined to the plane of the leaf at an angle$=60^\circ$, the perpendicular distance of point C from the plane of the paper will be z, the height of the right angle from the ground to be calculated.) In triangle ACB, $x=c$ cos. A (1). In triangle ACD, $y=x$ sin. A (2). Substituting in (2) the value of x in (1), we have $y=c$ cos. A sin. A (3). Now the right-angled triangle ACB is inclined to the horizon at an angle$=60^\circ$

$\therefore z=y$ sin. $60^\circ=c$ cos. A sin. A sin. 60° by substituting the value of y from (3), whence $z=41{\cdot}48$. (See Ex. p. 31.)

PROB. 22 (Fig. 25).

(1.) In triangle DBC, given DC$=100$, D$=33^\circ\ 45'$, and DBC$=$ACB$-$D$=51^\circ-33^\circ\ 45'=17^\circ\ 15'$, to find log. BC.

(2.) In triangle CBE, given log. BC, angle BCE=ACB—ACE=11°, and angle CEB=A+ACE=90°+40°=130°, to find BE, the required height.

PROB. 23 (Fig. 26).

(1.) In triangle ACD, given CD and all the angles, to find AD. (2.) In triangle BDE, given DE and all the angles, to find DB. (3.) In triangle ADB, given AD, DB, and included angle ADB, to find the side AB, the required distance.

PROB. 24 (Fig. 27).

(1.) In triangle ABD, given the three sides, to find the angle ABD, and thence its supplement ABO. (2.) In triangle ACB, given the three sides, to find the angle CAB and its supplement BAO. (3.) In triangle BAO, given BA and all the angles, to find the sides BO and AO, the required distances.

PROB. 25 (Fig. 28).

(1.) In triangle BAO, given AB and all the angles, to find log. AO. (2.) In right-angled triangle OAC, given angle OAC and log. AO, to find AC and OC.

PROB. 26 (Fig. 29).

(1.) In oblique-angled triangle ACB, given AC and all the angles, to find log. AB. (2.) In right-angled triangle ABH, given angle BAH and log. hypothenuse, to find height BH.

PROB. 27 (Fig. 29).

Let BH represent the tower, A the observer due south of tower, AC a straight line drawn due east, or at right angles to AH. (1.) In ABH are given BH=100 feet, and angle of elevation BAH=50°, to find AH=83·9. (2.) In right-angled triangle ACH, right-angled at A, are given AH and AC, to find CH=311·6. (3.) In right-angled triangle

BCH, given BH and CH, to find angle of elevation BCH= 17° 47′ 45″.

PROB. 28 (Fig. 30).

Let A and E be the observers, C the balloon; then angle of elevation CAD=20°, horizontal angle NAD=45° (the bearing of C from A), and horizontal angle AED=11° 15′ (the bearing of C from E). (1.) In horizontal triangle ADE are given the side AE=4000 yards, AED=11° 15′, and DAE=180°−45°, to find log. AD. (2.) In vertical right-angled triangle ADC are given log. AD and CAD=20°, to find CD, the height of balloon.

PROB. 29 (Fig. 31).

(1.) In triangle BCD, given DC=18 feet, BCD=90°−40° =50°, and BDC=90°+37° 30′=127° 30′, to find log. BC. (2.) In right-angled triangle ABC, given log. BC and ACB =40°, to find height AB and distance AC of steeple.

PROB. 30 (Fig. 32).

In triangle BRS, given BR=54 feet, RBS=90°−4° 52′ =85° 8′, and angle BSR=4° 52′−4° 2′, to find log. RS. (2.) In right-angled triangle SAR, given log. RS, and ASR =NRS=4° 2′, to find AS the distance and height AR of hill, and hence AB, the height of top of building.

PROB. 31 (Fig. 33).

Let the perpendicular CD be the breadth of the river; then (1.) in triangle ABC are given AB=500 and all the angles, to find log. BC. (2.) In right-angled triangle DBC are given log. BC and angle B=79° 12′, to find perpendicular CD, the required breadth of river.

Or thus, $\frac{BC}{AB}=\frac{\text{sin. } A}{\text{sin. } C}=\frac{\text{sin. } A}{\text{sin. } (A+B)}$

$$\therefore BC=AB \,.\, \text{sin. } A \text{ cosec. } (A+B) \quad .\,.\, (1)$$

and DC=BC sin. B; substituting the value of BC from (1), we have DC=AB sin. A cosec. (A+B) sin. B.

PROB. 32 (Fig. 34).

Let AB be the sloping base=340 feet, AD, BF, horizontal lines in the vertical plane ACD, and CD the height of hill required; then angle CBF=46° 15′, CAD=40° 10′, and EBF=EAD=10° 5′.

∴ CBE=CBF−EBF=36° 10′, and CAE=CAD−EAD =30° 5′ ∴ ACB=6° 5′.

(1.) In triangle ACB, given AB and all the angles, to find log. AC.

(2.) In right-angled triangle ACD, given log. AC and CAD, to find CD=1221; adding height of eye 5 feet, the height of hill=1226 feet.

PROB. 33 (Fig. 35).

Angles observed with a Theodolite, p. 81.

Let A be an object which we may suppose on the top of a hill, and B and C two stations on its sloping side. Conceive a horizontal plane to pass through the lowest station C, and let A*a* B*b* be perpendiculars on that plane, meeting it in *a* and *b*; join the points A, B, C in the *oblique* plane and the points *a*, *b*, C in the *horizontal* plane, and draw BD perpendicular to A*a*; then A*a*C, B*b*C will be right-angled triangles, right-angled at *a* and *b*, and BD*ab* a parallelogram.

There are given, BC the distance between the stations =220 yards, horizontal angle *b*C*a*=62° 33′ (subtended at C by A and B), horizontal angle C*ba*=70° 15′ (subtended at B by A and C), and the vertical angles AC*a*, BC*b*, the elevations of the object A and the upper station B, to find the horizontal distances *ba*, C*a*, and the heights A*a*, AD of A above C and B.

(1.) In right-angled triangle B*b*C, given BC=220 and BC*b*=8° 32′ ∴ perpendicular B*b*=32·6 and base C*b*=217·6. (2.) In horizontal triangle *ab*C, given side C*b*=217·6, the angle *ab*C=70° 15′ and *a*C*b*=62° 33′, to find *ab*=263·1 and *a*C=279·1. (3.) In the right-angled triangle A*a*C, given the side *a*C=279·1 and AC*a*=32° 12′, to find

A*a*=175·7. (4.) And since D*a*=B*b*=32·6 .·. AD=A*a* —B*b*=143·1.

PROB. 34 (last Fig.).

Angles observed with a Sextant, p. 81.

The same construction being made as in last problem; (1.) In right-angled triangle BC*b*, given BC=220 yards and BC*b*=8° 32′, to find B*b*=D*a*=32·6, and C*b*=217·6. (2.) In triangle ABC, given BC=220, and the three angles, .·. log. AC=2·518244, and AB=299·5. (3.) In right-angled triangle AC*a* are given log. AC and angle AC*a*=32° 12′ .·. A*a*=175·7, and C*a*=279·1. (4.) In right-angled triangle ABD, AD=A*a*—B*b*=175·7—32·6=143·1, and AB=299·5 .·. *ba*=BD=263·1.

PROB. 35 (Fig. 36).

(1.) In triangle CAD are given CD=549·4, C=57°, and CDA=14° .·. CAD=109° and AD=487·27. (2.) In triangle ABD are given AD=487·27 and the three angles; hence AB=349·52, and BD=498·7.

PROB. 36 (Fig. 37).

(1.) In triangle AFC are given CA=3000, C=80° 10′, and CFA=25° 8′, to find FA=6959. (2.) In triangle ABF are given FA=6959, AB=2000, and included angle FAB=34° 10′; hence FB=5422, the distance of fort from battery.

PROB. 37 (Fig. 38): see Ex. p. 91.

Describe a circle passing through the three points A, B, D, and join AE, EB. Then (by Geometry) since angles in the same segment are equal,

.·. EAB=EDB=15° 30′, and EBA=ADE=12° 15′.

(1.) In triangle AEB, the side AB=5·35, and angle EAB=15° 30′, and EBA=12° 15′. .·. AE=2·438.

(2.) In triangle ABC, the three sides are given, to find CAB=35° 24′, and CAE (=CAB—EAB)=19° 54′.

(3.) In triangle CAE, the side AC=2·8, AE=2·438, and angle CAE=19° 54′. ∴ angle ACE=58° 33′.

(4.) In triangle ACD, the side AC=2·8, ADC=12° 15′, and ACD=58° 33′.

∴ AD=11·26, CD=12·46, and CAD=109° 12′.

Lastly. In triangle ABD, ADB=27° 45′, BAD (=CAD −CAB)=73° 48′, and AB=5·35. ∴ BD=11·03.

Prob. 38 (Fig. 39): see Ex. p. 93.

Describe a circle passing through points A, B, D, as in last problem; then angle ADC=128° 40′ ∴ ADE=51° 20′; join AE, EB; then EAB=EDB=91° 20′−51° 20′=40°, and EBA=ADE=51° 20′.

(1.) In triangle AEB, given side AB=267, angle EAB= 40°, and BEA=88° 40′, to find BE=171·6.

(2.) In triangle ABC, the three sides are given to find ABC=92° 26′; hence the angle EBC=143° 46′.

(3.) In triangle CBE are given two sides and included angle, to find BCD=16° 16′ 30″; whence DBC=23° 43′ 30″.

(4.) In triangle BDC are given BC and all the angles, to find BD=91 and DC=131, the two distances required.

(5.) In triangle ADB are given AB=267, ABD=68° 42′ 30″ (=ABC−DBC), and ADB=91° 20′, to find AD=249, the third distance required.

Prob. 39 (Fig. 40).

Let A be the place of the ship when the headland C bears due north of her; join AC, and draw AB at right angles to AC; let B be the place of the ship when the headland bears north-west; draw BN parallel to AN, to represent the direction of the meridian, or north and south line; then CBN (the bearing of C from B)=4 points, or 45°. ∴ ∠ABC= 4 points, or 45°, and CAB is a right angle.

In right-angled triangle BAC are given AB=10 and angle BCA=45°; hence AC and BC may be found.

PROB. 40 (Fig. 41).

In triangle ACB, given angle C the course of the ship= 4 points or 45°, the angle A=45° (since B bears south-east from A), and side CB=14·5, to find the sides AC and AB.

PROB. 41 (Fig. 42).

In triangle ACB, the bearing of B from A is east, and of C from A is N.N.E. ∴ angle CAB=6 points=67° 30′; also angle ABC=4 points, and side BC=10; hence AB, the distance of boat from ship, can be found.

PROB. 42 (Fig. 43).

In triangle ACB, the bearing of C from A is N.E. b. N., and of B from A is E. b. N. ∴ angle CAB (the difference between E. b. N. and N.E. b. N.)=4 points, or 45°.

Again, the bearing of A from B is W. b. S., and of C from B is W.N.W. ∴ angle ABC=3 points, and side AB=15 miles; hence the other sides may be found.

PROB. 43.

In any triangle ABC, let AB=50; make the angle A=7 points, and angle B=1 point; then if B is due south of A, and C is to the westward of AB, the bearing of C from A is W. b. S., and from B, N. b. W.

In the triangle ABC are given the side AB and all the angles; hence the sides AC and CB may be found.

PROB. 44 (Fig. 44).

(1.) In triangle CAB, given AB=23 miles, CAB=$10\frac{1}{2}$ points, and CBA=$3\frac{1}{2}$ points ∴ CB=53·01. (2.) In triangle ABH, given AB=23 miles, ABH=8 points, and HAB=4 points ∴ AHB=4 points, and BH=AB=23 miles. (3.) In triangle CBH, given CB, BH, and included angle CBH= $4\frac{1}{2}$ points, to find CH=42·33, the distance of cape from headland; and BCH=$2\frac{1}{4}$ points. Hence C bears from H $2\frac{1}{4}$ points to the left, *i. e.* $2\frac{1}{4}$ points left of W.N.W. or W.$\frac{1}{4}$S.

Prob. 45.

This problem is similar to the last, the position and bearing of the bodies only being different. Make a figure to suit the case (roughly or by scale), by drawing from a point A the line AC W.N.W., the line AB N.W., and AH due north. Let AB=6, and through B draw BC W. b. S., and BH N.E., and join CH.

(1.) In CAB, find CB=4·13. (2.) In ABH, find BH=6. (3.) Lastly, in triangle CBH are given two sides, and included angle CBH=13 points, to find CH=9·71 miles, and angle CHB=13° 40′.

Hence C bears from B 13° 40′ to the right, and therefore bearing of C from H is S. (45°+13° 40′) W., or S. 58° 40′ W.

Prob. 46 (Fig. 46).

Let A and C represent the stations on deck, H the object on shore; then angle A=3 points (the bearing of H from C), and angle H=2 points (the difference of bearings at the two stations). In the triangle, all the angles are given, and also the side AC=100; hence the distance AH and CH may be found.

Prob. 47 (Fig. 45).

Let A be the required position of the ship; then in triangle ABC, the bearing of C from B is E.N.E., and A from B is due south, ∴ the angle CBA=10 points. We have ∴ given, BC=960, AB=2000, and included angle ABC=10 points, to find the angle BAC=20° 32′; hence the bearing of church is N. 20° 32′ E.

Prob. 48 (Fig. 46).

Let A be the required position of a ship; then in triangle ACH, bearing of C from H is W. $\frac{1}{4}$S. and of A from H is S. b. E.$\frac{1}{2}$E. ∴ angle CHA=9$\frac{1}{4}$ points. In triangle ACH we have given, the two sides CH=4·23, and AH=2·3, and included angle CHA, to find angle HAC=4$\frac{1}{2}$ points; hence the bearing of C is $4\frac{1}{2}$ points to the left of H, that is, to the left of N.b.W.$\frac{1}{2}$W., and the required bearing is $\therefore$ W.N.W.

PROB. 49 (Fig. 47).

(1.) In horizontal triangle ADS, given AS$=2640$, SD$=800$, and angle ASD$=3$ points$=33^\circ\ 45'$, to find AD$=2024{\cdot}2$. (2.) In right-angled triangle ABD, AD$=2024{\cdot}2$, and ADB $=5^\circ\ 25'$, to find BA the height$=192$ yards.

PROB. 50 (Fig. 48).

In triangle ABC, the angle ACB (the difference of bearings of A and B from C) is 3 points, the bearing of A from B is due west, and of C from B is S.W.b.W. $\therefore$ angle ABC$=3$ points, and side AC$=10$ miles; hence BC may be found.

PROB. 51 (Fig. 49).

From D, the highest part of muzzle, draw DG parallel to axis AE of piece, then angle BDG$=$BCA, the angle required.

Through B draw a line perpendicular to axis, cutting DG in H, and AE in K; then in right-angled triangle BDH are given BD$=77{\cdot}25$ inches, and BH ($=$BK$-$DE)$=6{\cdot}22-4{\cdot}92=1{\cdot}3$ inches, to find angle BDG$=$BCA, the angle required.

PROB. 52 (Fig. 50).

(1.) In triangle AED . . . cot. $\text{AED}=\frac{\text{ED}}{\text{AD}}=\frac{50}{175}$

$\therefore$ AED$=74^\circ\ 3'\ 15''$, and AEB$=148^\circ\ 6'\ 30''=$angle of the tenaille.

(2.) In triangle BAK are given AB$=350$, BK$=100$, and included angle ABK($=90^\circ-74^\circ\ 3'\ 15''$)$=15^\circ\ 56'\ 45''$, to find AK$=255{\cdot}33=$AG, the line of defence.

(3.) Since the angle of a regular pentagon$=108^\circ=\text{HAH}_{,}+2\ \text{BAH}$ $\therefore$ $\frac{1}{2}\ \text{HAH}_{,}+\text{BAH}=54^\circ$, or $\frac{1}{2}\ \text{HAH}_{,}=54^\circ-15^\circ\ 56'\ 45''=38^\circ\ 3'\ 15''$ $\therefore$ $\text{HAH}_{,}=76^\circ\ 6'\ 30''$, the flanked angle.

(4.) To find curtain FG. In triangle AED, $AE=\sqrt{AD^2+DE^2}=\sqrt{175^2+50^2}=182$; hence GE or EF $=AG-AE=255{\cdot}33-182=73{\cdot}33$.

Then, in isosceles triangle GEF, are given two sides, GE and EF, and included angle GEF=148° 6′ 30″, to find side GF=141·02, the curtain.

(5.) In triangle FEH are given EH=82, EF=78·33, and included angle HEF=31° 53′ 30″ (supplement of 148° 6′ 30″), to find angle EHF=63°; hence its supplement AHF=117° 0′, the angle of the shoulder.

(6.) In triangle FEH are given EH and all the angles to find FH=43·5, the length of flank.

(7.) And flanked angle $GFH=GFE+EFH=GFE+(AHF-HEF)=15°\ 56'\ 45''+85°\ 6'\ 30''=101°\ 3'\ 15''$.

The problems following that are marked with the letter *a* cannot easily be solved without some knowledge of algebra and *analytical* trigonometry.

PROB. 53 *a*.

In fig. 21, suppose DB=6 × AB; and angle ACB double of angle D. (1.) In triangle ADB . . . $\tan. D=\frac{AB}{DB}=\frac{AB}{6.AB}=\frac{1}{6}$ $\therefore D=9°\ 28'$, and angle ACB=18° 56′. (2.) In ACB . . . $\frac{CB}{AB}=\cot. ACB \therefore CB=AB\ .\ \cot. ACB$.

To find the numerical value of cot. ACB, or, as it is called, the natural cot. ACB,

log. nat. cot. ACB=log. tab. cot. ACB−10 . . . (p. 25)
=10·464672−10=·464672

$\therefore$ nat. cot. ACB=2·9 and CB=2·9 times the height AB.

PROB. 54 *a* (Fig. 51).

Let AC represent the part standing the first time, AE the part standing the second, DC the part broken the first time, and EF the part broken the second. Then AD=20, AF=30, and EC=5.

Let $AE=x$, $CD=y$; then $AC=x+5$ and $EF=y+5$, and $x+y+5=$height of maypole required.

In FAE and ACD, $x^2+30^2=(y+5)^2$ (1), and $(x+5)^2+20^2=y^2$ (2). By (1) $x^2+900=y^2+10y+25$. By (2) $x^2+10x+425=y^2$. Subtracting (1) from (2) we get $x+y=45$

$\therefore x+y+5=50$, the height.

PROB. 55 *a* (Fig. 52).

Let BD the horizontal distance of A from $B=x$, and CD the horizontal distance of A from $C=y$, AD the height of the $\text{hill}=z$.

Let elevation of A at $B=\alpha=20°$, and horizontal angle BCD (the bearing of A from $C)=\beta=67° \; 30'$, and $\theta=ACD$, the elevation of A from C required.

In right-angled triangle BCD, $x=y \sin. \beta$ (1). In vertical triangle ADB, $z=x \tan. \alpha$ (2). In vertical triangle ADC, $z=y \tan. \theta$ (3) $\therefore y \tan. \theta=x \tan. \alpha=y \sin. \beta \,.\, \tan. \alpha$

$\therefore \tan. \theta=\sin. \beta \,.\, \tan. \alpha$, and $\theta=18° \; 36'$.

PROB. 56 *a* (Fig. 53).

(1.) In the triangle ABC, given the three sides, to find one of the angles at the base, as B; and thence in the right-angled triangle ADB, the perpendicular AD is easily found.

Or thus:

Let $AD=x$; and $BD=y$; a, b, c the three sides of triangle; then $a=40$, $b=30$, and $c=20$. In right-angled triangle ABD, $x^2+y^2=c^2$ (1). In right-angled triangle ADC, $x^2+(a-y)^2=b^2$ (2). Subtracting (1) from (2), $(a-y)^2-y^2=b^2-c^2 \therefore y=13{\cdot}75$, and $x=\sqrt{c^2-y^2}=14{\cdot}523$, the perpendicular required.

PROB. 57 (Fig. 54).

(1.) In ABC are given all the angles and side AB, to find AC.

(2.) In ADC are given all the angles and side AC, to find DC.

Or thus:

Let DAC$=\alpha=23°\ 50'\ 15''$; CAB$=\beta=93°\ 4'\ 20''$; ABC$=\gamma=54°\ 28'\ 30''$; side AB$=\alpha=416$; AC$=x$; and DC$=y$.

(1.) In ABC, $\frac{x}{a}=\frac{\sin.\gamma}{\sin.\text{ACB}}=\frac{\sin.\gamma}{\sin.(\beta+\gamma)}$. (2.) In ADC, $y=x\tan.\alpha$ $\therefore$ $y=\alpha\sin.\gamma\operatorname{cosec.}(\beta+\gamma)\tan.\alpha=278{\cdot}7$.

Prob. 58 (Fig. 52).

Let dist. BD of first station$=x$; dist. CD of second station$=y$; height AD of tower$=z$; ABD$=\alpha=23°\ 18'$, ACD$=\beta=21°\ 16'$, and side BC$=300$. In ABD, $z=x\tan.\alpha$; in ACD, $z=y\tan.\beta$, $\therefore$ $x\tan.\alpha=y\tan.\beta$ and $\frac{x}{y}=\frac{\tan.\beta}{\tan.\alpha}$; and in BCD, $\frac{x}{y}=\cos.$ BDC, $\therefore$ $\cos.$ BDC$=\frac{\tan.\beta}{\tan.\alpha}=\tan.\beta.\cot.\alpha$, and $\therefore$ BDC$=25°\ 20'\ 45''$. Then in BDC, BD$=$BC cot. BDC$=633{\cdot}4$, and AD$=$BD $\tan.\alpha=272{\cdot}7$.

Prob. 59.

In fig. 20, let AC represent the pole, inclined at an angle A$=80°$, B the place of spectator; then AB$=100$ feet, and angle B$=54°$. In ABC are given side AB and all the angles, to find AC, the length of pole.

Prob. 60 *a* (Fig. 55).

Let AE (height of man)$=a=6$; AB (height of column) $=a+b=200$; BC (height of statue)$=c=50$. Let $\theta=$ADE or BDC, the angle subtended at D, the place of spectator, by a or c. Let $x=$BDE, and $y=$AD, the width of river required. In triangles AED, ABD, and ACD, we have $\frac{a}{y}=\tan.\theta$, $\frac{a+b}{y}=\tan.(\theta+x)$; $\frac{a+b+c}{y}=\tan.(2\theta+x)=\tan.(\overline{\theta+x}+\theta)$

$$=\frac{\tan.\ (\theta+x)+\tan.\ \theta}{1-\tan.\ (\theta+x)\ .\ \tan\ \theta}=\frac{\frac{a+b}{y}+\frac{a}{y}}{1-\frac{a+b}{y}\cdot\frac{a}{y}}=\frac{(2a+b)\ y}{y^2-(a+b)a}$$

$\therefore \frac{a+b+c}{y^2}=\frac{2a+b}{y^2-(a+b)\ a}$ substituting the values of a, b, c, we have $y=82\cdot5$ feet, the width of river.

PROB. 61 a (Fig. to Prob. 19).

Let AB$=x$, AC$=y$, AD$=z$.

In CAB, $y : x :: \sin.\text{ABC} : \sin.\ \text{ACB}$

$\therefore y=\frac{x\ \sin.\ \text{ABC}}{\sin.\ \text{ACB}}=x\ \sin.\ \text{ABC}\ .\ \text{cosec. ACB}=\cdot8465x$ (see Ex. p. 33).

In ABD, $z : x :: \sin.\ \text{ABD} : \sin.\ \text{ADB}$

$\therefore z=x\ \sin.\ \text{ABD cosec. ADB}=1\cdot1852x.$

In CAD, let CD$=a=6594$, let $b=1\cdot1852$ and $c=\cdot8465$; then $z=bx$, and $y=cx$, and angle CAD$=85°\ 46'=$A, are given, to find x or AB.

In ACD, $a=\frac{\sqrt{4yz\ \text{hav. A}}}{\sin.\ \theta}$ where $\tan.\ \theta=\frac{\sqrt{4yz\ \text{hav. A}}}{z-y}$ (Trig. Part II.); substituting the values cx and bx for y and z, we have $\tan.\ \theta=\frac{\sqrt{4cxbx\ \text{hav. A}}}{bx-cx}=\frac{\sqrt{4bc\ \text{hav. A}}}{b-c}$ whence θ may be found; and $a=\frac{\sqrt{4cxbx\ \text{hav. A}}}{\sin.\ \theta}\ \therefore\ x=\frac{a\ \sin.\ \theta}{2\sqrt{bc\ \text{hav. A}}}$ whence x or AB is found.

This useful problem may be solved without haversines thus: In triangle ACD $\frac{z+y}{z-x}=\frac{\tan.\ \frac{1}{2}\ (\text{ACD}+\text{ADC})}{\tan.\ \frac{1}{2}\ (\text{ACD}-\text{ADC})}$ (p. 53); or, $\frac{bx+cx}{bx-cx}=\frac{b+c}{b-c}=\frac{\tan.\ \frac{1}{2}(\text{ACD}+\text{ADC})}{\tan.\ \frac{1}{2}(\text{ACD}-\text{ADC})}$ whence the angles may be found, and thence the distance x, or AB$=4694$ yards.

PROB. 62 (Fig. 56).

Let height of tower, BC$=x$; flagstaff, CD$=4=a$; distance of observer, AB$=100=b$; DAC$=48'\ 20''=\alpha$, and BAC$=\theta$, the angle subtended by tower at A.

In BAC, $\frac{x}{b}=\tan.\theta$. In BAD, $\frac{x+a}{b}=\tan.(\theta+\alpha)=$ $\frac{\tan.\theta+\tan.\alpha}{1-\tan.\theta\tan.\alpha}=\frac{x+b\tan.\alpha}{b-x\tan.\alpha}$ by substitution. By solving the quadratic equation $\frac{x+a}{b}=\frac{x+b\tan.\alpha}{b-x\tan.\alpha}$ we have,

$$x=-\frac{a}{2}\pm\sqrt{\frac{a^2}{4}+\frac{ab-b^2\tan.\alpha}{\tan.\alpha}}$$ from which x may be found.

The calculation of tan. α is as follows:

$$\log.\tan.\alpha=\text{tab. }\log.\tan.\alpha-10\ (\text{p. }26).$$
$$=8{\cdot}148001-10=\bar{2}{\cdot}148001\ \therefore\ \tan.\alpha={\cdot}01406.$$

Substituting this value of tan. α and of a and b in the above equation,

$$x=-2+\sqrt{4+\frac{400-10000\times{\cdot}01406}{{\cdot}01406}}=133{\cdot}8\text{ yds}=401{\cdot}4\text{ ft}.$$

(See also the logarithmic calculation of Prob. 80, which is similar to the above.)

Or thus, by *circular measure:*

In problems of this kind, where one of the angles concerned is small (only a few minutes), the circular measure of the angle may be sometimes used with advantage (see Trig. Part II. p. 107), thus: Let CD (height of flagstaff)$=a=4$; AB (distance of spectator)$=b=100$; BC (height of tower)$=x$, and angle DAC$=\theta$.

With center A and distance AC describe the arc CE, cutting AD in E; then triangle ECD, being small, may be considered as a plane triangle, right-angled at E, and equiangular to the triangle BAC. Now circular measure of angle

$$\text{DAC}=\frac{\text{arc}}{\text{rad.}}=\frac{48'\ 20''}{57^{\circ}{\cdot}29577}=\frac{\text{EC}}{\text{AC}}\ (\text{Part II. p. 108})$$

but EC$=a$ cos. ECD$=a$ cos. BAC, and AC$=b$ sec. BAC

$$\therefore \frac{\text{EC}}{\text{AC}}=\frac{a \text{ cos. BAC}}{b \text{ sec. BAC}}=\frac{a}{b}\text{cos.}^2 \text{ BAC}=\frac{4}{100} \cdot \text{cos.}^2 \text{ BAC}$$

$$\therefore \text{ cos.}^2 \text{ BAC}=\frac{100 \times 48' \ 20''}{4 \times 57^\circ{\cdot}29577}=\frac{25 \times 0^\circ{\cdot}80555}{57^\circ{\cdot}29577}=\frac{25 \times 80555}{5729577}$$

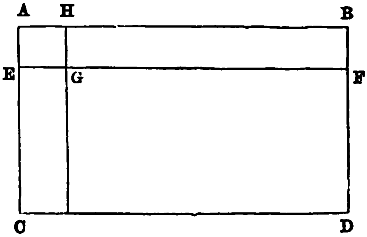
whence BAC$=53^\circ \ 38' \ 30''$, and $x=100 \times \text{tan. BAC}=407{\cdot}5$.

Prob. 63.

Since the area of a parallelogram is equal to the product of its base by its altitude,* and the area of triangle$=\frac{1}{2}$ area of a parallelogram with same base and altitude, $\therefore$ area triangle$=\frac{1}{2}$ base × alt.$=\frac{1}{2} \times 40 \times 30=600=66\frac{2}{3}$ square yards.

*** Proof that area AD=AB.AC.**

(1.) Let each of the sides AB, AC of the rectangular parallelogram AD contain AH or AE, the linear unit, an exact number of times; that is, let AB and AC be *commensurable* with AH.



Let the linear unit AH or AE$=x$, and the square AG=superficial unit.

Let AB contain a linear units$=ax$
AC . . . b . . . . $=bx$

Then it may be shewn that the *area* or number of superficial units in AD$=ab$, or area AD$=ab$ times AG.

$$\text{For (Eucl. VI. 1) } \frac{\text{AF}}{\text{AG}}=\frac{\text{AB}}{\text{AH}}=\frac{ax}{x}=a$$

$$\frac{\text{AD}}{\text{AF}}=\frac{\text{AC}}{\text{AE}}=\frac{bx}{x}=b$$

$$\therefore \text{ multiplying } \frac{\text{AD}}{\text{AG}}=ab$$

or AD$=ab$ times AG.

(2.) If the sides AB, AC be *incommensurable* with AH, another unit may be found which is commensurable with certain lines that approach *as near as we please to* AB *and* AC, and therefore the product of the units in such lines will represent the area of a rectangle differing from the area AD by less than any assignable quantity; that is, we may in this case also express the area of AD by the product of the number of units in AB and AC.

PROB. 64 *a*.

Let x=base of triangle cut off, y=perpendicular; then by note to last problem $xy=48$; and $x:y::12:16$ $\therefore$ $4x=3y$ $\therefore$ $x=6$ and $y=8$ and hypothenuse$=\sqrt{8^2+6^2}=10$.

PROB. 65 *a*.

Let x=side of equilateral triangle ABC. Then (Trig. Part II. p. 49) area$=\sqrt{s\,.\,\overline{s-a}\,.\,\overline{s-b}\,.\,\overline{s-c}}$, where $s=\frac{1}{2}(a+b+c)=\frac{3x}{2}$ since $a=b=c=x$ $\therefore$ area$=\sqrt{\frac{3x}{2}\cdot\frac{x}{2}\cdot\frac{x}{2}\cdot\frac{x}{2}}=\frac{\sqrt{3}}{4}\,.\,x^2$, or $180=\frac{\sqrt{3}}{4}x^2$ $\therefore$ $x=20\cdot389$.

PROB. 66 *a*.

In any triangle ABC (fig. 20), given base AB and angles A and B, to find area. (1.) Find one of the sides AC, and then, by Formula I. Trig. Part II. p. 49, the area. Or thus: In triangle ABC, $\frac{\text{AC}}{\text{AB}}=\frac{\sin.\text{B}}{\sin.\text{C}}=\frac{\sin.\text{B}}{\sin.(\text{A}+\text{B})}$ $\therefore$ AC=AB . sin. B cosec. (A+B). Now area$=\frac{1}{2}$. AB . AC . sin. A$=\frac{1}{2}\text{AB}^2$. sin. A sin. B cosec. (A+B)=33680.

PROB. 67 *a* (Fig. 20).

In triangle ABC, let $a=5$, $b=3$, and x=third side; then area$^2=s\,.\,\overline{s-a}\,.\,\overline{s-b}\,.\,\overline{s-x}=36$, where $s=\frac{1}{2}(a+b+x)=\frac{1}{2}(8+x)$ (Trig. Part II. p. 49) $\therefore$ $36=\left(\frac{8+x}{2}\right).\left(\frac{8+x}{2}-5\right).\left(\frac{8+x}{2}-3\right).\left(\frac{8+x}{2}-x\right)=\frac{(64-x^2).(x^2-4)}{16}$, whence x (the third side)=4 or $\sqrt{52}$ by solving quadratic.

PROB. 68 *a* (Fig. 57).

By the question, 2.AEF=ABC. Let AD, alt. of triangle$=a=10$, base BC$=x$, EF$=y$, the base of triangle AEF, and z=its alt.; then area AEF$=\frac{1}{2}$ABC $\therefore \frac{yz}{2}=\frac{1}{2}\cdot\frac{ax}{2} \therefore yz=\frac{1}{2}ax$ (1). But by similar triangles $\frac{z}{y}=\frac{a}{x} \therefore y=\frac{zx}{a}$. Substituting the value of y in (1), $\frac{z^2x}{a}=\frac{ax}{2} \therefore z=\frac{a}{\sqrt{2}}=\frac{10}{\sqrt{2}}=7{\cdot}071$, whence distance from base$=10-7{\cdot}071=2{\cdot}929$.

Prob. 69 *a*.

Let x=the less side, and y=the greater side of rectangle; then $2x$=greater side of triangle, and y its hypothenuse $\therefore \sqrt{y^2-4x^2}$=third side of triangle; also $2x$=side of square whose area equals that of the two allotments $\therefore 4x^2=xy+x\sqrt{y^2-4x^2}$ . . . (1). Length of paling for rectangle and triangle$=2(x+y)+2x+y+\sqrt{y^2-4x^2}$, and length of paling for square$=8x \therefore 8x+55=2(x+y)+2x+y+\sqrt{y^2-4x^2}$ . . (2). From equation (1) $y=\frac{5x}{2}$ which substituted in (2) gives $x=11 \therefore y=27{\cdot}5$; hence area rectangle$=xy=302{\cdot}5$, and area triangle$=x\sqrt{y^2-4x^2}=181{\cdot}5$.

Prob. 70 *a*.

Let $x+y$, x, and $x-y$ represent the three sides of the right-angled triangle; then $(x+y)^2=x^2+(x-y)^2$ and $\frac{1}{2}(x-y)x=216$. From these equations the values of x and y are found, and thence the sides of triangle.

Prob. 71 *a*.

Let side of equilateral triangle$=x \therefore$ area$=\frac{1}{2}x^2$ sin. $60=\frac{1}{2}x^2\cdot\frac{1}{2}\sqrt{3}$ (Trig. Part II. p. 49) $=\frac{x^2}{4}\sqrt{3} \therefore$ expense of paving $=8.\frac{x^2}{4}\sqrt{3}=2x^2\sqrt{3}$, and expense of palisading three sides$=84\times 3x=252x \therefore 2x^2\sqrt{3}=252x$, and $x=72{\cdot}74$.

PROB. 72 *a.*

Let AB, one side of a regular polygon (as fig. p. 110, Part II.), $=a$, number of sides$=n$ $\therefore$ angle AOB$=\frac{1}{n}\,.\,360^\circ$ and AOK$=\frac{1}{n}\,.\,180^\circ$. Area polygon$=n\times$area triangle AOB $=2\,n\times$area triangle AOK$=2\,n\,.\,\frac{1}{2}$ AK . KO$=n$. AK . KO; but KO$=$AK cot. AOK$=$AK . cot. $\frac{180^\circ}{n}$ $\therefore$ area polygon$=$ $n\,.\,\text{AK}^2\,.\,\cot.\,\frac{180^\circ}{n}=\frac{na^2}{4}\cot.\,\frac{180^\circ}{n}$ since AK$=\frac{a}{2}$

Ex. If $n=8$ and $a=16$ yards $\therefore$ area$=512$ cot. $22^\circ\ 30'=$ 1236·1 yards.

PROB. 73 *a.*

In last problem, area $\text{A}=\frac{na^2}{4}\cot.\,\frac{180^\circ}{n}$

$$\therefore\ a^2=\frac{4\text{A}}{n\cot.\,\frac{180^\circ}{n}}=\frac{4\text{A}}{n}.\tan.\,\frac{180^\circ}{n}$$

$\therefore$ side $a=\sqrt{\frac{4\text{A}}{n}.\tan.\,\frac{180^\circ}{n}}=\sqrt{\frac{4\times 1236{\cdot}1}{8}.\tan.\,22^\circ\ 30'}=$ 16 in Ex. 1.

PROB. 74 *a* (Fig. 58).

The area of triangle ABC$=\frac{1}{2}$. AB . AC . sin. BAC (Part II. p. 49)$=\frac{1}{2}n$. AD . AC . sin. BAC (1); since n . AD$=$AB.

$$\text{In triangle AED, }\frac{\text{AE}}{\text{AD}}=\frac{\text{sin. ADE}}{\text{sin. AED}}=\frac{\text{sin. BAC}}{\text{sin. BAG}}$$

$\therefore$ AE sin. CAG$=$AD sin. BAC.

Substituting this value of AD sin. BAC in (1), we have area ABC$=\frac{1}{2}n$AC . AE sin. CAG; but AC . AE$=\text{AF}^2$, since AF is a mean proportional to AC and AE.

$\therefore$ Area ABC$=\frac{1}{2}n\text{AF}^2$. sin. CAG

$=n$. area triangle AFH$=$area polygon.

PROB. 75 (Fig. 60).

Let CB represent length of ship$=160$, A place of observer; then $\text{CAB}=20' \, 15''$; also the angles SBC, SCE are nearly equal to each other, since the side CB (subtended by an angle of a few minutes) is small compared with AB or AC.

$\therefore$ either of the angles SBC or SCE may be considered to be the course$=22° \, 30'$ (Nav. Part I. p. 2).

In triangle ABC, $\dfrac{\text{AB}}{\text{BC}}=\dfrac{\sin. \text{C}}{\sin. \text{A}}=\dfrac{\sin. 22° \, 30'}{\sin. 20' \, 15''}$

$\therefore \text{AB}=\text{BC} . \sin. 22° \, 30' \text{ cosec. } 20' \, 15''=10247$ feet$=1{\cdot}94$ miles.

Otherwise (by circular measure).

Problems involving small angles may sometimes be solved more simply, by employing the circular measure of the angle (Trig. Part II. p. 110).

Let AD=AB, join DB, then (the angle A being small) each of the angles ABD, ADB is nearly equal to a right angle; and DB may be considered as the arc subtending the angle A. Now circular measure of angle $\text{A}=\theta=\dfrac{\text{arc}}{\text{rad.}}=\dfrac{\text{DB}}{\text{AB}}=\dfrac{20' \, 15''}{57°{\cdot}29577}$

$\therefore \text{AB}=\text{DB} . \dfrac{57°{\cdot}29577}{20' \, 15''}$ but $\text{DB}=\text{BC} . \sin. \text{BCD}$ (considering CDB as a right-angled triangle)$=160 . \sin. 22° \, 30'$.

$\therefore \text{AB}=\dfrac{160 . \sin. 22° \, 30' \times 57{\cdot}29577 \times 60 \times 60}{1215 \times 1760 \times 3}$ (by reducing to seconds, and dividing by 1760×3 to obtain distance in miles)$=1{\cdot}94$ miles.

PROB. 76 *a* (Fig. 61).

Let AB represent the meridian, angle $\text{A}=x°$, angle $\text{B}=x°+10°$; then C is the point where the ships meet, and the perpendicular $\text{CD}=100$ miles (the distance from the meridian AB). The sides AC and BC are to each other as 3 : 2. Let $\text{AC}=3y \;\therefore\; \text{BC}=2y$.

Then (Trig. Part I. p. 53) $\dfrac{3y+2y}{3y-2y}=\dfrac{\tan. \frac{1}{2}(x+10+x)}{\tan. \frac{1}{2}(x+10-x)}$

or, $5=\frac{\tan.(x+5)}{\tan.5}$.·. tan. $(x+5)=5\tan.5$.·. $x=18^\circ\ 37'\ 45''$ and difference of lat. AB=AD+DB=CD cot. x+CD cot. $(x^\circ+10^\circ)=479\cdot8$.

Prob. 77 *a* (Fig. 62).

Let AB represent the height=2 yards, AA′ the diameter of the earth; draw the tangent BD; then BD will be the distance seen by spectator (neglecting the effects of refraction, &c.). By Geometry, $BD^2=BA\,.\,BA'=2\times14080000$ (if we suppose diameter=8000 miles=14080000 yards)

.·. BD=5307 yards.

Prob. 78 *a* (Fig. to Prob. 77).

Let BH be drawn perpendicular to BA, the height of mountain; then HBD (the angle of depression of D, a point in the horizon)=90°−CBD=DCB, since angle D=90°
.·. DCB=2° 13′ 27″. Let a=BA=3 miles; x=CA or CD, the radius of the earth; then $\sec. C=\frac{CB}{CD}=\frac{x+a}{x}=1+\frac{a}{x}$

$$.\cdot.\ \frac{a}{x}=\sec.C-1=\frac{1}{\cos.C}-1=\frac{1-\cos.C}{\cos.C}=\frac{2\sin.^2\frac{C}{2}}{\cos.C}$$

.·. $2x=a\cos.C\operatorname{cosec.}^2\frac{C}{2}$.·. $2x$, or diameter=7952 miles.

Prob. 79 *a* (Fig. to Prob. 77).

Let DC or AC, the radius of the earth=a=4000 miles; BA, height of mountain=x. Then $\frac{x+a}{a}=\sec.C=\sec.1^\circ\ 31'$;

$$\text{or, } \frac{x}{a}=\sec.C-1=\frac{1-\cos.C}{\cos.C}=\frac{2\sin.^2\frac{C}{2}}{\cos.C}$$

$$.\cdot.\ x=2a\sin.^2\frac{C}{2}\cos.C=1\cdot402 \text{ miles.}$$

Prob. 80 *a* (Fig. 63).

First solution (geometrically).

Upon the line $AB = 400$ yards, the distance between the objects A and B, describe a segment of a circle containing an angle equal to the given angle ($35^\circ\ 10'$) (Euclid III. 33), and measure a perpendicular distance $CD = 560$ yards, the width of the river; through D draw EDF parallel to base AB, cutting the circle in E and F; then either of the points E, F will correspond to the station on the other side of the river; and the lines EA, EB, or FA, FB, measured on the same scale as AB, will be the required distances.

Second solution (trigonometrically).

(1.) In right-angled triangle AGC (G being the center of the circle), given $AC = 200$, and angle $AGC = AEB = 35^\circ\ 10'$, to find radius $AG = 347{\cdot}75$, and $GC = 283{\cdot}87$, and angle $CAG = 54^\circ\ 50'$.

(2.) $\therefore GD = CD - GC = 560 - 283{\cdot}87 = 276{\cdot}13$.

(3.) In right-angled triangle EDG are given $GD = 276{\cdot}13$, radius $EG = 347{\cdot}75$, to find angle $EGD = 37^\circ\ 19'\ 30''$.

(4.) Then the angle $AGE = 180^\circ - (AGC + EGD) = 180^\circ - (35^\circ\ 10' + 37^\circ\ 19'\ 30'') = 107^\circ\ 30'\ 30''$.

(5.) Draw GH perpendicular to AE. We have now given AG and angle $AGH = 53^\circ\ 45'\ 15''$ in right-angled triangle AGH, to find $AH = \frac{1}{2}AE$, whence $AE = 560{\cdot}1$.

(6.) To find the other side, EB, there are given AE and AB, and included angle $EAB = CAG + GAH = 54^\circ\ 50' + 36^\circ\ 14'\ 45'' = 91^\circ\ 4'\ 45''$, to find $EB = 694{\cdot}37$.

Third solution (analytically).

By the geometrical construction of the fig. (see first solution) we find that the two objects are on the same side of the station. Let A be the nearest object to station, B the more distant.

AB (distance of objects)$= a = 400$; produce BA to K, meeting perpendicular from station E, and let $AK = x$, $AE = y$, $BE = z$, angle $AEK = \theta$, angle AEB between objects$= \alpha = 35^\circ\ 10'$, and perpendicular $EK = b = 560$.

In triangle AEK, $\frac{x}{b}=\text{tan. }\theta$

In triangle BEK, $\frac{a+x}{b}=\text{tan. }(\theta+\alpha)=\frac{\text{tan. }\alpha+\text{tan. }\theta}{1-\text{tan. }\alpha\text{ tan. }\theta}$

$$\therefore a+x=\frac{b\text{ tan. }\alpha+b\text{ tan. }\theta}{1-\text{tan. }\alpha\text{ tan. }\theta}$$

$$=\frac{b\text{ tan. }\alpha+x}{1-\text{tan. }\alpha\frac{x}{b}}=\frac{b^2\text{ tan. }\alpha+bx}{b-x\text{ tan. }\alpha}$$

whence (by substituting the above value of tan. θ)

$$x^2+ax=\frac{ab-b^2\text{ tan. }\alpha}{\text{tan. }\alpha}$$

and solving quadratic, $x=-\frac{a}{2}\pm\sqrt{\left\{\frac{a^2}{4}+\frac{ab-b^2\text{ tan. }\alpha}{\text{tan. }\alpha}\right\}}$

To find numerical value of this expression:

First find b^2 tan. α.

log. b............2·748188

log. b............2·748188

tan. α............9·847913

5·344289

$\therefore b^2$ tan. α=220947

and ab=224000

$\therefore ab-b^2$ tan. α=3053

log. 3053............3·484727

log. cot. α..........0·152087

3·636814

$$\therefore \frac{ab-b^2\text{ tan. }\alpha}{\text{tan. }\alpha}=4333$$

$$\frac{a^2}{4}=40000$$

$$\therefore \frac{a^2}{4}+\frac{ab-b^2\text{ tan. }\alpha}{\text{tan. }\alpha}=44333$$

and $\sqrt{44333}=210{\cdot}5$ $\therefore x=-200\pm210{\cdot}5=10{\cdot}5$ (using the upper sign; the lower sign giving a corresponding point to the right of B). With this value of x we find AE=560·1, and EB=694·3.

PROB. 81 *a* (Fig. 64).

Let AB represent the tower, CD the road, C the place of spectator; then angle ACB=20° (the elevation of tower at C), and ACD=30° (the angle between line joining the top and road CD). Let BD be the nearest distance of tower from road (=200 feet); then plane of triangle ABD is perpendicular to road, and therefore ADC is a right angle.

In triangle ABC, AB=AC sin. ACB=AC sin. 20°

In triangle ADC, AD=AC sin. ACD=AC sin. 30°

$$\therefore \frac{AB}{AD}=\frac{\sin. 20^\circ}{\sin. 30^\circ} \quad \text{but } \frac{AB}{AD}=\sin. ADB$$

$$\therefore \sin. ADB=\frac{\sin. 20^\circ}{\sin. 30^\circ} \quad \text{or } ADB=43^\circ\ 9'\ 15''$$

then in triangle ABD, AB=DB tan. ADB

$$=200 . \tan. 43^\circ\ 9'\ 15''=187{\cdot}5 \text{ feet.}$$

Or thus (by Spherical Trigonometry).

The angle at C is a solid angle, formed by the vertical angle ACB, the oblique angle ACD, and horizontal angle BCD which call θ; let ACB=α, ACD=β. To find the horizontal angle θ we may consider the point C the center of a sphere, and the three angles, α, β, θ, may then be represented by the three sides of a right-angled spherical triangle (Part II. p. 60), of which α and θ contain the right angle, and β is the hypothenuse.

By Rule XIII. Part I., $\cos. \beta=\cos. \alpha \cos. \theta$;

$$\text{whence } \cos. \theta=\cos. \beta \sec. \alpha,$$
$$=\cos. 30^\circ \sec. 20^\circ=22^\circ\ 50'\ 15'';$$

then in right-angled triangle BDC are given BD and angle BCD, to find CB, which with the angle ACB will give AB, the height of tower.

PROB. 82 *a*.

Let $7x$ and $4x$ represent the sides, and included angle= 129° 34′; let θ and φ denote the other two angles.

$$\therefore \theta+\varphi=180^\circ-129^\circ\ 34'=50^\circ\ 26'.$$

$$\text{Now } \frac{7x+4x}{7x-4x}=\frac{\tan. \frac{1}{2}(\theta+\varphi)}{\tan. \frac{1}{2}(\theta-\varphi)} \quad . . . \text{ (Trig. Part II. p. 49.)}$$

$$\text{or } \frac{11}{3}=\frac{\tan. 25^\circ\ 13'}{\tan. \frac{1}{2}(\theta-\varphi)} \quad \therefore \tan. \tfrac{1}{2}(\theta-\varphi)=\frac{3}{11} \tan. 25^\circ\ 13'$$

which determines $\theta-\varphi$; and therefore with $\theta+\varphi$ already known, the angles θ and φ may be found.

PROB. 83 *a*.

First solution.

Let θ, 2θ, 3θ denote the angles; x, y, z the sides opposite to them respectively;

$$\text{then } x+y+z=6 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\theta+2\theta+3\theta=180^\circ \quad \therefore\ \theta=30^\circ$$

$$\text{Now } \frac{x}{z}=\frac{\text{sin. } \theta}{\text{sin. } 3\theta}=\frac{\text{sin. } 30^\circ}{\text{sin. } 90^\circ}=\tfrac{1}{2} \quad \therefore\ 2x=z \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\text{and } \quad \frac{x}{y}=\frac{\text{sin. } \theta}{\text{sin. } 2\theta}=\frac{\text{sin. } 30^\circ}{\text{sin. } 60^\circ}=\frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}}=\frac{1}{\sqrt{3}}$$

$$\therefore\ x\sqrt{3}=y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Substituting in (1) the values of y and z in (3) and (2), we have

$$x+x\sqrt{3}+2x=6, \text{ or } x=\frac{6}{3+\sqrt{3}}=3-\sqrt{3}=1{\cdot}268$$

$$\text{whence } z=2x=2{\cdot}536, \text{ and } y=x\sqrt{3}=2{\cdot}196$$

Second Solution.

This problem, and problems 84, 85, and 96, in which are given (or may be found) the perimeter and all the angles, can be solved in the following manner.

Let ABC be the triangle (fig. 20).

To find side a. By (C), (D), Part. II. p. 49,

$$b\,c \text{ sin.}^2\frac{A}{2}=\tfrac{1}{2}(a+b-c)\ .\ \tfrac{1}{2}(a-b+c)$$

$$a\,c \text{ cos.}^2\frac{B}{2}=\tfrac{1}{2}(a+c+b)\ .\ \tfrac{1}{2}(a+c-b)$$

$$a\,b \text{ cos.}^2\frac{C}{2}=\tfrac{1}{2}(a+b+c)\ .\ \tfrac{1}{2}(a+b-c)$$

$$\therefore\ \frac{a^2b\,c \text{ cos.}^2\frac{B}{2} \text{ cos.}^2\frac{C}{2}}{b\,c \text{ sin.}^2\frac{A}{2}}=\frac{\frac{1}{4}(a+b+c)^2.\frac{1}{2}(a+c-b).\frac{1}{2}(a+b-c)}{\frac{1}{2}(a+b-c)\ .\ \frac{1}{2}(a-b+c)}$$

$$\therefore\ a=\tfrac{1}{2}(a+b+c)\ .\text{ sin. }\frac{A}{2}\text{ sec. }\frac{B}{2}\text{ sec. }\frac{C}{2}=1{\cdot}268.$$

PROB. 84 *a*.

Let θ, $2\,\theta$, $4\,\theta$ denote the angles; x, y, z the sides opposite to them respectively; then $x+y+z=100$...(1)

$$\theta+2\,\theta+4\,\theta=180^\circ \;\therefore\; \theta=25^\circ\ 42'\ 51''.$$

Now $\dfrac{z}{y}=\dfrac{\sin.\ 4\,\theta}{\sin.\ 2\,\theta}=\dfrac{2\sin.\ 2\,\theta\ .\ \cos.\ 2\,\theta}{\sin.\ 2\,\theta}=2\cos.\ 2\,\theta=1{\cdot}247$

$\therefore z=1{\cdot}247y.$ $\dfrac{y}{x}=\dfrac{\sin.\ 2\,\theta}{\sin.\ \theta}=\dfrac{2\sin.\ \theta\ .\ \cos.\ \theta}{\sin.\ \theta}=2\cos.\ \theta=1{\cdot}802$

$\therefore x=\dfrac{y}{1{\cdot}802}$ Substituting in (1) we have $\dfrac{y}{1{\cdot}802}+y+1{\cdot}247y=100$ $\therefore y=35{\cdot}69$ $\therefore x=19{\cdot}8$, and $z=44{\cdot}51$.

Otherwise, see Prob. 83, second solution.

PROB. 85 *a*.

Let x, y, z denote the base, perpendicular, and hypothenuse respectively; then $x+y+z=24$, $x=z\cos.\ 30^\circ=\frac{1}{2}\sqrt{3}\,.z$, $y=z\sin.\ 30^\circ=\frac{1}{2}z$ $\therefore (\frac{1}{2}\sqrt{3}+\frac{1}{2}+1)z=24$, and $z=10{\cdot}144$ $\therefore x=8{\cdot}784$, and $y=5{\cdot}072$.

Otherwise, see Prob. 83, second solution.

PROB. 86 *a* (Fig. 20).

Let the two sides AC and AB be denoted by x and y; then $x+y=600$, $a=400$, and $A=80^\circ$, are given to solve the triangle. By (D) Part II. p. 49, we have $xy\ .\ \cos.^2\dfrac{A}{2}=\frac{1}{2}(x+y+a)\ .\ \frac{1}{2}(x+y-a)$ $\therefore xy\ .\cos.^2\ 40^\circ=\frac{1}{2}(600+400)\ .\ \frac{1}{2}(600-400)=500\ .\ 100$ $\therefore xy=50000\ .\ \sec.^2\ 40^\circ=85204.$ Hence, knowing the values of $x+y$ and xy, the sides x and y may be found.

PROB. 87 *a*.

By (C) Part II. p. 49, $bc\ \sin.^2\dfrac{A}{2}=\frac{1}{2}(a+b-c)\ .\ \frac{1}{2}(a-b+c)$

where $a+b=600$, $c=400$, and $A=60^\circ$ $\therefore \sin.^2\dfrac{A}{2}=\sin.^2\ 30^\circ=\frac{1}{4}$ $\therefore 400\ .\ \frac{1}{4}\ .\ b=\frac{1}{2}(200)\ \frac{1}{2}(a-b+400)$ $\therefore 400b=200(a-b)+80000$ $\therefore 2b=a-b+400$, or $3b-400=a=600-b$ $\therefore 4b=1000$ $\therefore b=250$, whence $a=350$.

PROB. 88 *a*.

Let hypothenuse and perpendicular be denoted by x and y; then $x+y+8=24$, and $x^2=y^2+64$ $\therefore$ $x=10$ and $y=6$.

PROB. 89 (Fig. 21).

Let AB represent the steeple, C the first station, D the second station; then CB=80, and DB=250. Let angle ACB$=2\alpha$, then angle D$=\alpha$ $\therefore$ angle DAC$=\alpha$, and AC$=$ DC$=170$.

$$\therefore \text{AB}=\sqrt{\text{AC}^2-\text{CB}^2}=\sqrt{170^2-80^2}=150.$$

PROB. 90 *a*.

Let AB, AC, the sides of a triangle ABC, be denoted by x and y; then we have given angle A, side a, and $x+y=b$, to solve the triangle. By (D) Part II. p. 49,

$$xy \cos.^2\frac{\text{A}}{2}=\tfrac{1}{2}(x+y+a)\,.\,\tfrac{1}{2}(x+y-a)=\tfrac{1}{4}(b+a)\,.\,(b-a)$$

$\therefore$ $xy=\frac{1}{4}(b+a)\,.\,(b-a)\,.\,\sec.^2\frac{\text{A}}{2}$. From this equation, and $x+y=b$, the values of x and y may be found.

PROB. 91 *a*.

Let AB, AC, the sides of a triangle ABC, be denoted by x and y; then we have given the angle A, side a, and $x-y=d$, to solve the triangle.

By (C) Part II. p. 49,

$$xy \sin.^2\frac{\text{A}}{2}=\tfrac{1}{2}(a+x-y)\,.\,\tfrac{1}{2}(a-\overline{x-y})$$
$$=\tfrac{1}{4}(a+d)\,.\,(a-d)$$
$$\therefore\ xy=\tfrac{1}{4}(a+d)\,.\,(a-d)\,.\,\text{cosec.}^2\frac{\text{A}}{2}$$

From this equation, and $x-y=d$, the values of x and y may be found.

PROB. 92 *a*.

Let A denote the angle opposite base a, X and Y the angles at the base, and x and y the sides opposite to X and Y. Then $x+y=b$, and $X-Y=d$. Now $\frac{\sin. X}{\sin. A}=\frac{x}{a}$, and

$$\frac{\sin. Y}{\sin. A}=\frac{y}{a};\ \text{adding,}\ \frac{\sin. X+\sin. Y}{\sin. A}=\frac{x+y}{a}=\frac{b}{a}$$

$$\text{or}\ \frac{\sin. X+\sin. Y}{\sin. (X+Y)}=\frac{b}{a}\ \text{or}\ \frac{2\sin.\frac{1}{2}(X+Y)\cos.\frac{1}{2}(X-Y)}{2\sin.\frac{1}{2}(X+Y)\cos.\frac{1}{2}(X+Y)}=\frac{b}{a}$$

$$\therefore \frac{\cos.\frac{1}{2}d}{\cos.\frac{1}{2}(X+Y)}=\frac{b}{a}\ \therefore \cos.\tfrac{1}{2}(X+Y)=\frac{a}{b}\cos.\tfrac{1}{2}d.$$

whence $X+Y$ may be found, and ∴ with $X-Y=d$, the values of X and Y.

Prob. 93 *a*.

Let a be the base, and x and y the other sides of the triangle; X, Y the angles opposite to x and y respectively, and A the angle opposite a.

Then $x-y=d$, and $X-Y=D$.

$$\frac{\sin. X}{\sin. A}=\frac{x}{a},\ \text{and}\ \frac{\sin. Y}{\sin. A}=\frac{y}{a};\ \text{subtracting,}\ \frac{\sin. X-\sin. Y}{\sin. A}$$

$$=\frac{x-y}{a}=\frac{d}{a}\quad \therefore \frac{2\cos.\frac{1}{2}(X+Y)\sin.\frac{1}{2}(X-Y)}{\sin. (X+Y)}=\frac{d}{a},$$

$$\text{or}\ \frac{2\cos.\frac{1}{2}(X+Y)\sin.\frac{1}{2}(X-Y)}{2\sin.\frac{1}{2}(X+Y)\cos.\frac{1}{2}(X+Y)}=\frac{d}{a}\quad \therefore \sin.\tfrac{1}{2}(X+Y)$$

$=\frac{a}{d}\cdot\sin.\frac{1}{2}(X-Y)=\frac{a}{d}\sin.\frac{1}{2}D$, and since $X-Y$ is also known, the sides and angles may be found.

Prob. 94 *a*.

Let ABC (fig. 20) be any triangle; then angle A, side b, and the difference $(a-c=d)$ between a and c are given, to solve the triangle. By (C) and (D) Part II. p. 49, $bc\sin.^2\frac{A}{2}$ $=\frac{1}{2}(a+b-c)\cdot\frac{1}{2}(a-b+c)$ and $bc\cdot\cos.^2\frac{A}{2}=\frac{1}{2}(b+c+a)\cdot$ $\frac{1}{2}(b+c-a)$ ∴ $\tan.^2\frac{A}{2}=\frac{(a+b-c)\cdot(a+c-b)}{(b+c+a)\cdot(b+c-a)}$. Similarly,

$$\tan.^2\frac{C}{2}=\frac{(c+b-a)\,.\,(c+a-b)}{(b+c+a)\,.\,(b+a-c)}\ \therefore\ \frac{\tan.^2\frac{C}{2}}{\tan.^2\frac{A}{2}}=\left(\frac{b+c-a}{a+b-c}\right)^2$$

$$\therefore\ \tan.\frac{C}{2}=\frac{b+c-a}{a+b-c}\,.\,\tan.\frac{A}{2}=\frac{b-(a-c)}{b+(a-c)}\,.\,\tan.\frac{A}{2}=\frac{b-d}{b+d}\,.\,\tan.\tfrac{1}{2}A.$$

PROB. 95 *a*.

Let ABC (fig. 20) be any triangle; then angle A, side b, and the sum $(a+c=m)$ of a and c are given, to solve the triangle. As in last problem, $\tan.^2\frac{A}{2}=\frac{(a+b-c)\,.\,(a+c-b)}{(b+c+a)\,.\,(b+c-a)}$,

$$\tan.^2\frac{C}{2}=\frac{(c+b-a)\,.\,(c+a-b)}{(b+a+c)\,.\,(b+a-c)}\ \therefore\ \tan.\frac{A}{2}\,.\,\tan.\frac{C}{2}=\frac{a+c-b}{a+b+c}$$

$$=\frac{m-b}{m+b}\ \therefore\ \tan.\frac{C}{2}=\frac{m-b}{m+b}\,.\,\cot.\frac{A}{2}.$$

PROB. 96 *a*.

See Problem 83, second solution.

PROB. 97 *a*.

In fig. 21, produce BCD to a point E, and join AE. Let AB represent the object, and C, D, E the three stations.

$DC=a$, $ED=b$, the angle $ACB=2\theta$, $ADC=90°-\theta$, and $AED=\theta$.

The triangle AEC is manifestly isosceles, since angle $ACB=AEC+EAC$, or $2\theta=\theta+EAC$, $\therefore$ $EAC=\theta$, or $AC=EC=a+b$.

In AEC, $\frac{AE}{a+b}=\frac{\sin.\ 2\theta}{\sin.\ \theta}$ $\quad\therefore\ AE=(a+b)\cdot\frac{\sin.\ 2\theta}{\sin.\ \theta}$.

In AED, $\frac{AE}{b}=\frac{\sin.\ (90°-\theta)}{\sin.\ (90°-2\theta)}=\frac{\cos.\ \theta}{\cos.\ 2\theta}\ \therefore\ AE=\frac{b\cos.\ \theta}{\cos.\ 2\theta}$

$\therefore\ \frac{b\cos.\ \theta}{\cos.\ 2\theta}=\frac{(a+b)\,.\,\sin.\ 2\theta}{\sin.\ \theta}$ or $\cos.\ 2\theta=\frac{b}{2\,(a+b)}$. In ACB,

$CB=(a+b)\cos. 2\theta \therefore CB=\frac{1}{2}b$; hence $AB^2=AC^2-CB^2=(a+b)^2-(\frac{1}{2}b)^2 \therefore$ height $AB=\sqrt{(a+b+\frac{1}{2}b).(a+b-\frac{1}{2}b)}=\sqrt{(a+\frac{3}{2}b).(a+\frac{1}{2}b)}$.

If $a=b=20$, then the height$=\sqrt{50\times 30}=38\cdot73$.

Prob. 98 *a* (Fig. 38).

Let the angles BDC, ADC be denoted by α and β respectively, and the sides of the triangle ABC by a, b, c, in the usual manner. Let angle $CAD=x$, and $CBD=y$; then in triangles BDC and ADC, we have

$$\frac{a\sin. y}{\sin. \alpha}=CD=\frac{b\sin. x}{\sin. \beta}\quad \therefore \frac{\sin. x}{\sin. y}=\frac{a\sin. \beta}{b\sin. \alpha}$$

$$\text{or}\quad \frac{\sin. x+\sin. y}{\sin. x-\sin. y}=\frac{a\sin. \beta+b\sin. \alpha}{a\sin. \beta-b\sin. \alpha}$$

$$\text{(Part II. p. 33)}\quad \frac{\tan. \frac{1}{2}(x+y)}{\tan. \frac{1}{2}(x-y)}=\frac{a\sin. \beta+b\sin. \alpha}{a\sin. \beta-b\sin. \alpha}$$

$$\text{whence } \tan. \tfrac{1}{2}(x-y)=\frac{1-\dfrac{b\sin. \alpha}{a\sin. \beta}}{1+\dfrac{b\sin. \alpha}{a\sin. \beta}}.\tan. \tfrac{1}{2}(x+y)$$

and $x+y$, the sum of the angles, is known, since it is$=360^\circ-C-\alpha-\beta$, and by the equation $x-y$ is given, hence the angles x and y may be found.

Applying this expression to an example, suppose that $\alpha=54^\circ\ 13'\ 45''$, $\beta=33^\circ\ 52'$, $C=77^\circ\ 42'\ 24''$, and that $\log. a=4\cdot170262$, $\log. b=4\cdot021189$.

$$\text{First find the value of } \frac{b\sin. \alpha}{a\sin. \beta}\text{, which is } 1\cdot0329,$$

$$\text{then } \tan. \tfrac{1}{2}(x-y)=\frac{-0\cdot0329}{2\cdot0329}.\tan. \tfrac{1}{2}(x+y)$$

$$\text{or } \tan. \overset{+}{\tfrac{1}{2}}(x-y)=\frac{-329}{20329}\overset{-}{\tan.}\ 97^\circ\ 5'\ 55''$$

(placing the proper sign over tan. 97° 5′ 55″, and thus determining the sign of tan. $\frac{1}{2}(x-y)$, (see Rule, Trig. Part I. p. 30).

$$\begin{array}{rrrr} \text{whence } \frac{1}{2}(x-y)= & 7^\circ & 24' & 24'' \\ \text{and } \frac{1}{2}(x+y)= & 97 & 5 & 55 \\ \hline \therefore\ x= & 104 & 30 & 19 \\ y= & 89 & 41 & 31 \end{array}$$

All the angles of the triangles ACD and CDB being now known, it will be easy to calculate the distances AD, CD, BD. If we perform the operation, we shall find AD=12516, CD=18241, and AD=10742·5.

Otherwise :

This problem is more easily solved by using a subsidiary angle, thus: assume $\tan.\ \theta=\frac{b \sin.\ \alpha}{a \sin.\ \beta} \ldots (1)$

$$\text{then } \tan.\ \tfrac{1}{2}(x-y)=\frac{1-\tan.\ \theta}{1+\tan.\ \theta}\,.\,\tan.\ \tfrac{1}{2}(x+y)$$

$$=\tan.\ (45^\circ-\theta)\,.\,\tan.\ \tfrac{1}{2}(x+y)$$

Find θ from equation (1), which is $45^\circ\ 55'\ 45''$, whence $\frac{1}{2}(x-y)=7^\circ\ 25'$, and the values of x and y may then be found as before.

PROB. 99 *a*.

Let straight lines OA, OB, OC, drawn at right angles to one another, form at the point O the solid angle of a cube.

Let $OA=a$, $OB=b$, $OC=c$; join the extremities A, B, C, and let the sides of the triangle ABC be denoted by a_1, b_1, c_1;

$$\text{then area ABC}=\tfrac{1}{2}b_1c_1 \sin.\ A=\tfrac{1}{2}b_1c_1\sqrt{1-\cos.^2 A}$$

$$=\tfrac{1}{2}b_1c_1\sqrt{1-\left(\frac{b_1^2+c_1^2-a_1^2}{2b_1c_1}\right)^2}=\tfrac{1}{4}\sqrt{4b_1^2c_1^2-(b_1^2+c_1^2-a_1^2)^2}$$

Now in right-angled triangles AOC, AOB, and BOC, we have $b_1^2=a^2+c^2$, $c_1^2=a^2+b^2$, $a_1^2=b^2+c^2$,

$\therefore b_1^2+c_1^2-a_1^2=2a^2$ and $4b_1^2c_1^2=4(a^2+c^2)\,.\,(a^2+b^2)$ $=4(a^4+a^2b^2+a^2c^2+b^2c^2)$; substituting these values, we have

$$\text{area ABC}=\tfrac{1}{2}\sqrt{a^2b^2+a^2c^2+b^2c^2}.$$

PROB. 100 *a*.

Let ABC be an equilateral triangle, and upon the sides describe squares, the centers of which are A′, B′, C′; join the points A′, B′, C′, forming the equilateral triangle A′B′C′, A′ being opposite A, &c., then area A′B′C′=area ABC . $(1+\frac{1}{2}\sqrt{3})$.

For area $A'B'C'=\frac{1}{2}B'C'^2.\sin. 60^\circ$, area $ABC=\frac{1}{2}AC^2.\sin. 60^\circ$

$$\therefore \text{area } A'B'C'=\frac{B'C'^2}{AC^2}\,.\,\text{area } ABC.$$

Now to find the value of B′C′ in terms of AC.

$$\text{In triangle } A'B'C',\ 2AB'.AC'\cos. B'AC'=AB'^2+AC'^2-B'C'^2$$

$$\text{or } 2AB'^2\cos. 150^\circ=2AB'^2-B'C'^2$$

$$\therefore B'C'^2=2AB'^2(1-\cos. 150^\circ)=2AB'^2(1+\cos. 30^\circ)$$

$$=2AB'^2(1+\tfrac{1}{2}\sqrt{3})$$

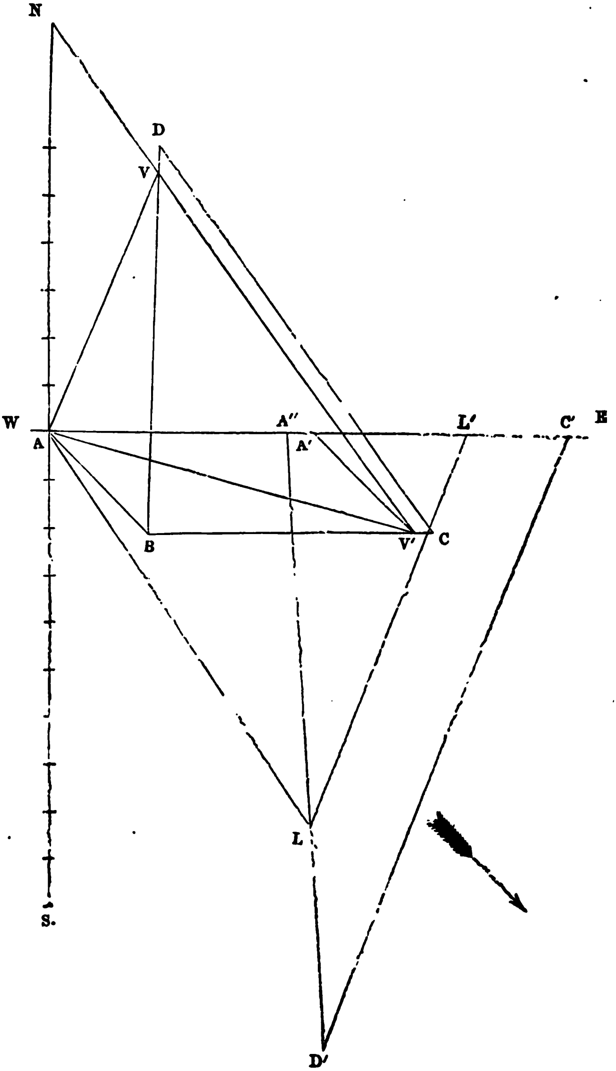
$$\text{But } AB'^2=2\,.\left(\frac{AC}{2}\right)^2=\tfrac{1}{2}AC^2 \;\therefore\; B'C'^2=AC^2(1+\tfrac{1}{2}\sqrt{3})$$

$$\text{hence area } A'B'C'=(1+\tfrac{1}{2}\sqrt{3})\,.\,\text{area } ABC.$$

PROB. 101 *a*.

The geometrical construction of this problem may be made as follows:

To find the course and distance of the vice-admiral. Draw N.S. and W.E. at right angles, and let A, the point of intersection, be the place of the admiral. Make the angle NAV=2 points, or N.N.E., and take from a scale AV=6; then V is the place of the vice-admiral. Make SAB=4 points, or S.E., and AB=3, then it will simplify the problem if we suppose the admiral to be transferred to B, and to sail due east along the line BV′, instead of along AE, which is the actual line described by the admiral; for then the problem will be reduced to this: to find the course and distance of the vice-admiral, so as to form a junction with the admiral. Join BV, and take BC=6, the rate of admiral's steaming, and with C as a center and radius=10 (the rate of vice-admiral) describe an arc of a circle cutting BV (or BV produced) in


N
D
V
W
A″
L′
C′
E
A
A′
B
V′
C
L
S.
D′

D; join DC. Through V draw VV′ parallel to DC; then VV′ is the vice-admiral's distance, and the angle ANV which it makes with the meridian N.S. is the course.

$$\text{For } \frac{\text{rate of admiral}}{\text{rate of vice-ad.}}=\frac{BC}{DC}=\frac{BV'}{VV'}$$

∴ V′ is the point of junction, on the hypothesis that the admiral sailed from B. But the admiral steamed along the line AE, and described AA′=BV′ in the same time; ∴ A′ is the position of the admiral when the vice-admiral is at V′, and it is evident the position of V′ from A′ is the one required, namely, S.E. 3 miles.

To find the course and distance of the look-out steamer. Make the angle SAL=3 points or S.E.b.S., and AL=10 miles, then L is the position of the look-out with respect to the admiral. Take AA″=5 miles (the distance the steamer is to be a-head of the admiral), and suppose, as before, the admiral to be transferred to A″; join LA″; then the problem may be solved by supposing the steamer to form a junction with the admiral, as in the former case.

From A″E take A″C′=6 miles, the rate of admiral, and with C′ as center, and distance 14 miles, describe an arc cutting A″D′ in D′, and join D′C′. Through L, the place of the steamer, draw LL′ parallel to D′C′. Then LL′ is the steamer's distance run to get into her station, and the angle it makes with the meridian is her course.

$$\text{For } \frac{\text{rate of admiral}}{\text{rate of steamer}}=\frac{A''C'}{C'D'}=\frac{A''L'}{L'L}$$

∴ L′ is the point of junction of the admiral and steamer; but since we have supposed the admiral transferred to a point A″, when in fact he was at A, the position of admiral will be 5 miles to the west of the steamer.

To find the course and distance of reinforcement. Join AV′; then AV′ is the distance run, and the angle V′AS the course of reinforcement.

This problem may be solved trigonometrically, thus:

To find the course and distance of vice-admiral.

(1.) In triangle AVB, given AV=6, AB=3, and included angle VAB=10 points=112° 30′, to find the other angles; hence angle ABV=46° 18′ 30″, and AVB=21° 11′ 30.″

(2.) In the triangle BDC are given BC=6, DC=10, and the angle DBC=180°−45°−46° 18′ 30″=88° 41′ 30″, to find angle BDC=BVV′=36° 51′ 30″

adding AVB=21 11 30

∴ angle AVV′=58 3 0=5 points nearly.

∴ the course of the vice-admiral=5 points to the left of admiral's bearing, or 5 points to the left of S.S.W. ∴ course =S.E.b.S. nearly.

(3.) In triangle AVB are given AV=6, VAB=10 points or 112° 30′, and angle ABV=46° 18′ 30″, to find BV=7·67 miles.

(4.) In triangle VBV′ are given BV=7·67, angle VBV′ =88° 41′ 30″, and VV′B=54° 27′, to find VV′=9·42, the distance run by vice-admiral.

To find the course and distance of look-out steamer.

(1.) In triangle AA″L, given AL=10, AA″=5, and included angle A″AL=5 points or 56° 15′, to find the other angles; hence angle AA″L=93° 49′ 30″, and ALA″= 29° 55′ 30″.

(2.) In triangle A″D′C′ are given A″C′=6, C′D′=14, and angle D′A″C′=180°−93° 49′ 30″=86° 10′ 30″, to find angle A″D′C′=A″LL′=25° 19′ 0″

adding ALA″=29 55 30

∴ course of steamer=55 14 30 to the right of ad.
bearing of adm. N.W.b.N.=33 45 0 left of north

∴ course of look-out=21 29 30 right of north
or N.N.E. nearly.

(3.) In triangle ALA″ are given AL=10, angle AA″L= 93° 49′30″, and angle A″AL=5 points=56° 15′, to find the side LA″=8·33.

(4.) In triangle A″LL′ are given LA″=8·33, angle LA‴L =86° 10′ 30″, and A″L′L=68° 30′ 30″, to find LL′=8·94, the distance run by look-out.

To find the course and distance run of the reinforcement.

(1.) To find BV′ or AA′ the distance run by admiral.

BV′ : VV′(=9·42) : : 6 : 10 $\therefore$ BV′=5·62

Or thus: in triangle VBV′ are given all the angles and side VV′ to find BV′.

(2.) To find distance AV′.

In triangle ABV′ are given two sides, AB and BV′, and included angle ABV′, to find AV′=8 miles.

(3.) To find course of reinforcement.

In same triangle find angle BAV′=29° 44′ left of B.

bearing of B, S.E.=45 0 left of S.

bearing of reinforcement=74 44 left of S.

hence course of reinforcement=E.b.S.½S. nearly.

To find time of reaching their appointed stations.

10 : 9·42 : : 1 h. : x=56 min. for vice-admiral

14 : 8·92 : : 1 h. : x=38 min. for look-out.

To find the rate of steaming of reinforcement, we have

56m. : 1 h. : : 8 : z=8½ miles.

SECTION II.

PROBLEMS IN ASTRONOMY, &c.

PROB. 102 (Fig. 73).

LET NWSE represent the horizon, NZS the celestial meridian, WZE the prime vertical (see definitions, Trig. Part I. p. 123): take NP=50° 48′, the latitude of spectator; then P is the pole of the heavens, since the altitude of the pole=lat. of place (p. 126), and PZ is the colatitude; take PQ=90°, then Q is a point in the celestial equator: through E and W, the east and west points, and Q draw the circle WQE; this will represent the celestial equator (p. 125). Let X represent the place of the sun at the time of observation, and through X draw XZ a circle of altitude, and XP a circle of declination: then in the spherical triangle PXZ are given, the polar distance PX (=90°−decl.), the zenith distance XZ (=90° −altitude), and the colatitude of spectator PZ (=90°−lat.), to find the angle PZX, the azimuth of the sun (p. 127). Then by Rule VIII. p. 62, PZX=111° 51′ (see Ex. p. 130). Hence the azimuth is N. 111° 51′ W., the angle being measured from the north towards the west.

PROB. 103 (Fig. 73).

The same construction being made as in the preceding problem, there are given the three sides of the triangle PZX, to find the hour-angle of the sun ZPX, which is also the apparent time, since the heavenly body is west of meridian

(p. 127). Whence (Rule VIII.) ZPX=$2^h\ 57^m\ 16^s$, the hour-angle required.

PROB. 104 (Fig. 73).

In the fig. are given the polar dist. PX, the zenith distance ZX, and the azimuth PZX, to find the hour-angle P; this is easily done by Rule X. p. 68. Or thus:

$$\frac{\text{sin. P}}{\text{sin. az.}}=\frac{\text{sin. zen. dist.}}{\text{sin. pol. dist.}}=\frac{\text{cos. alt.}}{\text{cos. decl.}}$$

$$\therefore \text{sin. P}=\text{sin. az. . cos. alt. . sec. decl.}$$

Whence P=$2^h\ 57^m\ 16^s$ (see Ex. p. 31).

PROB. 105 (Fig. 73).

In the fig. are given, the polar distance PX=74° 0′, co-latitude PZ=39°12′, hour-angle P=38° 0′; to find the zenith distance ZX, and thence the altitude.

Calculation by Rule IX. p. 65; see Ex. p. 131.

PROB. 106.

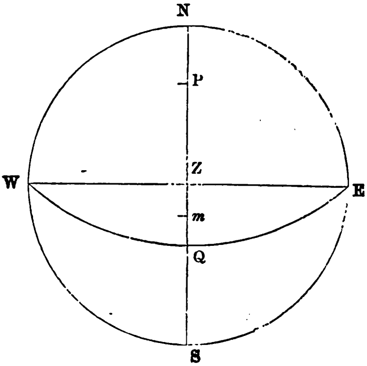
Construct a similar figure to the one in the preceding problem, but place the heavenly body on the east side of the meridian, since the hour-angle is A.M.

In the fig. are given, the polar dist. PX=77° 31′, colatitude PZ=39° 12′, hour-angle P=2h. 53m. 1s.; to find the angle PZX, the azimuth. Calculation by Rule XI. p. 69.

In the following twelve problems the heavenly body is supposed to be on the meridian at the time of observation; in this position the triangle PZX (fig. 73) vanishes, and the calculations are simplified.

PROB. 107.

Describe the horizon NWSE, the meridian NZS, and the prime vertical WZE: take Sm=70°, to represent the sun's meridian altitude: then m is the place of the sun. (The altitude is measured from the south point of the horizon S, and not from the north, since by the problem the zenith Z of the spectator is *north* of the sun.) And since the declination of the sun is 20° north, the equator must be drawn 20° to the south of the heavenly body: take therefore $mQ = 20°$, and draw the circle WQE, to represent the celestial equator. Then the latitude of spectator ZQ is found as follows:



$$\text{latitude} = ZQ = Zm + mQ = \text{zen. dist.} + \text{decl.}$$
$$= 90° - 70° + 20° = 40°$$

On the meridian take a point P, 90° distant from the equator, then P (the elevated pole) is the north pole of the heavens, and since Z, the zenith of the spectator, is by the figure north of the equator, the place is in north latitude, ∴ lat. = 40° N.

Prob. 108 (Fig. 66).

Describe the horizon NWSE, the meridian NZS, and prime vertical WZE. Take a distance Sm on the meridian (=70°) to represent the sun's meridian altitude: then m is the place of the sun. (The altitude is measured from S, the south point of the horizon, since the zenith of spectator is north.) And since the declination is *south* 5°, the equator must be drawn 5° north of the heavenly body: take, therefore, $mQ = 5°$, through Q draw WQE, the celestial equator.

$$\text{Lat.} = ZQ = Zm - mQ = \text{zen. dist.} - \text{decl.}$$
$$= 90° - \text{alt.} - \text{decl.} = 90° - 70° - 5° = 15°$$

The latitude is north, since the zenith is north of equator.

Note. In the diagrams fig. 66, &c., the distances are taken roughly; it being sufficient to indicate the form of the figure, and not the correct values of the quantities.

Prob. 109 (Fig. 67).

Describe as before the horizon, celestial meridian, and prime vertical. Take a distance Nm on the meridian$=70°$, to represent the star's meridian altitude; then m is the place of the star. (The altitude in this case is measured from N, the north point of the horizon, since the zenith of the spectator is by the problem south of the body.) And since the declination is 25° north, take $m\text{Q}=25°$, measured from m southward, and draw WQE, the celestial equator.

$$\text{Lat.}=\text{ZQ}=m\text{Q}-m\text{Z}=\text{star's decl.}-\text{zen. dist.}$$
$$=25°-20=5°$$

The latitude is north, since by the fig. the zenith is north of the equator.

Prob. 110 (Fig. 68).

Describe horizon, celestial meridian, and prime vertical. Take N$m=30°$, the sun's meridian altitude (measured from N, since the zenith is south): take $m\text{Q}=10°$, the sun's decl., then a circle through WQE will represent the celestial equator.

$$\text{Lat.}=\text{ZQ}=\text{Z}m-m\text{Q}=\text{zen. dist.}-\text{decl.}$$
$$=60°-10°=50°.$$

The latitude is south, since the zenith is, by the figure, south of the equator, and therefore the spectator is in the southern hemisphere.

Prob. 111 (Fig. to Prob. 107).

Describe the horizon, celestial meridian, and prime vertical. Since the latitude is 40° N. take $\text{ZQ}=40°$, and draw WQE to represent the celestial equator: from Q measure Q$m=20°$, then m is the place of the heavenly body, since its declination is 20° N., and

$$m\text{Z}=\text{zen. dist.}=\text{ZQ}-\text{Q}m=40°-20°=20°$$
$$\therefore \text{ altitude S}m=70°.$$

PROB. 112 (Fig. 68).

Describe horizon, celestial meridian, and prime vertical. Take $Nm=30°$, then m is the place of the sun (the altitude being measured from the north, since the zenith is south of the heavenly body) : take $ZQ=50°$=the latitude of place, and through Q draw WQE to represent the celestial equator (ZQ is taken to the north of Z, since the place is in the southern hemisphere): then mQ is the declination of body; and

$$mQ=mZ-ZQ=\text{zen. dist.}-\text{lat}=60°-50°=10°.$$

The declination is 10° north, since by the figure the heavenly body is north of the equator.

PROB. 113.

Describe the following parts of fig. 66, the horizon NWSE, celestial meridian NZS, and prime vertical WZE: then since the place is on the equator, the celestial equator must pass through the zenith, and therefore WZE represents also the celestial equator: take a distance $Sm=57°$, to represent the star's meridian altitude (measured from S, since the zenith must be north of the star); then

$$mZ=33°=\text{zenith dist.}=\text{declination},$$

since the equator passes through Z.

The declination is south, m being south of Z.

PROB. 114 (Fig. 69).

Let m be the place of the heavenly body below the pole P, and m_1 its place above pole.

Then Nm=star's meridian altitude at inferior transit.

Nm_1= . . . . . superior transit.

and $mP=m_1P$=star's polar distance.

$$\text{Now lat.}=NP=Nm+mP$$
$$\text{lat.}=NP=Nm_1-m_1P$$
$$\therefore\ 2\ \text{lat.}=Nm+Nm_1\ (\text{since } mP=m_1P)$$
$$\therefore\ \text{lat.}=\tfrac{1}{2}(Nm+Nm_1)=45°\text{N}.$$

PROB. 115 (Fig. 70).

Describe horizon and celestial meridian. Take Nm= 20°=star's meridian altitude at inferior transit; and since it passes through the zenith at superior transit, bisect Zm in P; then P is the pole, and (as in last problem)

$$\text{lat. PN}=\tfrac{1}{2}(\text{N}m+\text{NZ})=\tfrac{1}{2}(20^\circ+90^\circ)=55^\circ.$$

PROB. 116.

Construct the horizon NWSE, and celestial meridian NZS. From N take a meridian altitude=x, and from Z a meridian zenith distance=x;

$$\therefore \text{ meridian altitude at superior transit}=90^\circ-x$$
$$\therefore \text{ lat.}=\tfrac{1}{2}(\text{alt. below pole}+\text{alt. above pole})$$
$$=\tfrac{1}{2}(x+90^\circ-x)=45^\circ.$$

PROB. 117.

Describe the horizon NWSE, and celestial meridian NZS; on the meridian, take NP=60°, the latitude; then P is the pole; and since the declination is 40°, the polar distance=50°; take therefore from P towards N a distance Pm =50°, and from P towards the zenith a distance Pm_1=50°; then m is the place of the body below the pole, and m_1 is the place of body above the pole.

$$\text{Alt. below pole}=\text{N}m=\text{NP}-\text{P}m=60^\circ-50^\circ=10^\circ$$
$$\text{alt. above pole}=\text{S}m_1=90^\circ-Zm_1=90^\circ-(\text{P}m_1-\text{PZ})$$
$$=90^\circ-(50^\circ-30^\circ)=70^\circ.$$

PROB. 118.

In fig. 70, let mZ represent the parallel of declination described by the star. Then lat.=PN=50°48′
∴ PZ=39° 12′=star's polar distance. ∴ decl.=50° 48′ N. and alt. below pole=Nm=NP−Pm=50° 48′−39° 12′= 11° 36′.

PROB. 119.

Since the sun, in this problem, touches the horizon at the inferior transit, its polar distance must be equal to the latitude of the place (Trig. p. 126), hence lat.$=90^\circ-23^\circ\ 28'$ $=66^\circ\ 32'$ N.

Proof of Formulæ in p. 135.

(1). The heavenly body on the *prime vertical*, as at Y in fig. p. 125.

Draw the horizon, celestial meridian, and prime vertical; take a point Y on the prime vertical, and draw PY, the polar distance; then

cos. hour-angle=cot. lat.. tan. decl. (α)

Proof. In triangle PZY, right-angled at Z, mark the hour-angle P, polar distance PY (the co-decl.), and co-lat. PZ; then

cos. P=tan. PZ. cot. PY (Rule XIII. p. 72.)
or cos. hour-angle=cot. lat. . tan. decl.

Again, sin. decl.=sin. lat. . sin. alt. (β)

Proof. In a similar triangle, mark pol. dist. PY, co-lat. PZ, and zen. dist. YZ; then

cos. PY=cos. PZ . cos. YZ
or sin. decl.=sin. lat. . sin. alt.

Again, cos. alt.=sin. hour-angle. cos. decl. (γ)

Proof. In a similar triangle, mark zen. dist. YZ, hour-angle P, and co-decl. PY; then

sin. YZ=sin. P. sin. PY
or cos. alt.=sin. hour-angle . cos. decl.

Again, cos. lat.=cot. hour-angle . cot. alt. (δ)

Proof. In a similar triangle, mark zen. dist. YZ, hour-angle P, and co-lat. PZ; then

sin. PZ=cot. P . tan. YZ
or cos. lat.=cot. hour-angle . cot. alt.

(2.) The heavenly body *six hours*, or 90° from the meridian, as at V in fig. p. 125.

Draw the horizon, celestial meridian, and prime vertical, and take a point V six hours from the meridian, and draw PV the pol. dist., and ZV the zen. dist.; then

sin. alt.=sin. decl.. sin lat. (ε)

Proof. In triangle PVZ, right-angled at P, mark zen. dist. VZ, pol. dist. PV, and co-lat. PZ; then

cos. VZ=cos. PV. cos. PZ
or sin. alt.=sin. decl.. sin. lat.

Again, cos. azimuth=cot. lat.. tan. alt. (ζ)

Proof. In a similar triangle, mark PZV the azimuth, co-lat. PZ, and zen. dist. VZ; then

cos. Z=tan. PZ. cot. VZ
or cos. azimuth=cot. lat.. tan. alt.

Again, cos. lat.=cot. decl.. cot. azimuth (η)

Proof. in a similar triangle, mark co-lat., pol. dist., and azimuth; then

sin. PZ=tan PV. cot. Z
or cos. lat.=cot. decl.. cot. azimuth.

Again, cos. decl.=sin. azimuth. cos. alt. (θ)

Proof. In a similar triangle, mark pol. dist., azimuth, and zen. dist.; then

sin. PV=sin. Z. sin. VZ
or cos. decl.=sin. azimuth. cos. alt.

(3.) The heavenly body in the *horizon*, as at D, fig. p. 125.

Draw the horizon and celestial meridian as before, and take D a point on the horizon, and draw PD the polar distance, and ZD the zen. dist.; then

cos. hour-angle=−tan. lat.. tan. decl. (ι)

Proof. In quadrantal triangle PZD, mark hour-angle P, co-lat. PZ, and pol. dist. PZ; then (ZD being 90°)

cos. P=−cot. PZ. cot. PD
or cos. hour-angle=−tan. lat.. tan. decl.

Again, sin. decl.=cos. lat.. sin. amplitude. (κ)

Proof. In a similar triangle, mark PD, PZ, and Z the co-amplitude; then

$$\text{cos. PD}=\text{sin. PZ}\,.\,\text{cos. Z}$$
$$\text{or sin. decl.}=\text{cos. lat.}\,.\,\text{sin. amplitude.}$$

Again, sin. lat. $=-$cot. amplitude . cot. hour-angle (λ)

Proof. In a similar triangle, mark PZ, Z, and P; then

$$\text{cos. PZ}=-\text{tan. Z}\,.\,\text{cot. P}$$
$$\text{or sin. lat.}=-\text{cot. amplitude}\,.\,\text{cot. hour-angle.}$$

Lastly, cos. amplitude$=$cos. decl. . sin. hour-angle (μ)

Proof. In a similar triangle, mark Z, PD, and P; then

$$\text{sin. Z}=\text{sin. PD}\,.\,\text{sin. P}$$
$$\text{or cos. amplitude}=\text{cos. decl.}\,.\,\text{sin. hour-angle.}$$

In the above formulæ the lat. and decl. are supposed to be of the same name. When they are of different names, the last four formulæ (ι), (κ), (λ), (μ), may require some modification: this may be easily done by remembering that PD$=90°+$decl. instead of $90°-$decl., and angle Z$=90°+$ amplitude instead of $90°-$ampl., as in former case. The other formulæ require no alteration, since they are applied to problems in which the latitude and declination are always of the same name.

PROB. 120.

Describe the following parts of fig. p. 125, the horizon, celestial meridian, and prime vertical. Let Y be the place of the body on the prime vertical, Z the zenith of spectator, and P the pole; then in the spherical triangle PYZ, right-angled at Z, are given PY (the complement of the decl.) $=70°$, and YZ (the complement of alt.)$=60°$, to find PZ the complement of latitude, and thence the latitude.

By Rule XIII. p. 72,

$$\text{cos. PY}=\underline{\text{cos. PZ}}\,.\,\text{cos. YZ} \quad \text{(see Ex. p. 73).}$$
$$\therefore \text{PZ}=46°\ 50'\ 30'',\ \text{and lat.}=43°\ 9'\ 30''\ \text{N.}$$

PROB. 121.

Describe fig. to prob. 120. In this there are given, colat. PZ=90° – 50° 48′, or 39° 12′, pol. dist. PY=90° – 23°27′45″, or 66° 32′ 15″, to find zen. dist. YZ (and thence the alt.), and hour-angle P.

(1).	(2).
Mark PY and PZ as known quantities, and YZ as unknown; then, Rule XIII.	Mark PY and PZ as known quantities, and P as unknown; then
cos. PY=cos. PZ . cos. YZ	cos. P=cot. PY . tan. PZ
∴ YZ=59° 5′, and alt.=30° 55′	∴ hour-angle=4h. 37m. 4s.

If the student is acquainted with analytical trigonometry he can simplify the above and similar formulæ that will frequently occur in the following pages, before he opens the table of logarithms. Thus:

since cos. PY=cos. PZ . cos. YZ
∴ sin. decl.=sin. lat. . sin. alt.
hence sin. alt.=sin. decl. . cosec. lat.
cos. P.=cot. PY. tan. PZ
or cos. hour-angle=tan. decl. cot. lat.

From these two expressions the altitude and hour-angle are found directly from the quantities given in the question.

PROB. 122.

Make a right-angled triangle similar to PYZ in fig. p. 125, but so that Y may be to the east of meridian, somewhere on the line ZE; let Y be the place of the star whose altitude YE is 20°; take another point Y′ between Z and Y to represent the place of the other star, whose altitude is 40°, and join PY′; then in triangles PYZ, PY′Z, are given, colat. PZ=39° 12′, and zenith distances ZY (=70°) and ZY′ (=50°), to find the hour-angles at P, and thence YPY′ their difference.

In triangle ZPY
sin. PZ=tan. ZY . $\underline{\text{cot. ZPY}}$
whence ZPY=5h. 8m. 11s.

In triangle ZPY′
sin. PZ=tan. ZY′. $\underline{\text{cot. ZPY′}}$
and ZPY′=4h. 8m. 15s.

∴ difference of hour-angles=0h. 59m. 56s.

The above formulæ simplified:

$\underline{\text{cot. ZPY}}$=cos. lat.. tan. alt. $\underline{\text{cot. ZPY′}}$=cos. lat.. tan. alt.

PROB. 123.

Describe horizon, celestial meridian, and prime vertical; take P the pole, as in fig. p. 125, and make the angle VPZ= 6 hours, or 90°; let V be the heavenly body, and join ZV; then in triangle ZPV are given, ZV=90°—alt., PV=90° —decl., and P=90°, to find PZ=90°—lat.

By Rule XIII. cos. ZV=$\underline{\text{cos. PZ}}$. . cos. PV,
whence colat. PZ may be found, and thence the latitude.

Simplified . . . . sin. alt.=sin. lat.. sin. decl.
∴ sin. lat.=sin. alt.. cosec. decl.

PROB. 124.

Construct a similar figure to the one in the preceding problem; then in the right-angled triangle ZPV are given, colat. PZ=39° 12′, polar dist. PV=66° 32′ 15″, and P= 90°, to find azimuth Z, and zen. dist. ZV, and thence the altitude.

By Rule XIII. $\underline{\text{cos. ZV}}$=cos. PV. cos. PZ
sin. PZ=tan. PV. $\underline{\text{cot. Z}}$
Simplified . . . . . $\underline{\text{sin. alt.}}$=sin. decl.. sin. lat.
$\underline{\text{cot. az.}}$=cos. lat.. tan. decl.

PROB. 125.

Describe the following parts of fig. 97, horizon, celestial meridian, and prime vertical. Let M be the place of the sun at rising, and P the pole; join ZM and PM; then in quadrantal triangle ZPM are given, colat. PZ=39° 12′, azimuth

PZM=100°, and ZM=90°, to find hour-angle P, and thence apparent time (def. p. 127).

By Rule XIV. p. 75, $\overset{+}{\cos.\ PZ} = -\overset{-}{\tan.\ PZM}\,.\,\underline{\overset{+}{\cot.\ P}}$

placing the proper signs over known quantities, by Rule, p. 29, we find P is less than 90°, or 6 hours,

whence P is found=5h. 28m. 53s.

∴ app. time=18h. 31m. 7s., or 6h. 31m. 7s. A.M.

Simplifying formula . . cot. P=sin. lat. . cot. PZM
=sin. 50° 48′. cot. 80°.

Prob. 126.

In fig. p. 125, suppose D the place of the sun on the horizon; then amplitude=WD=37° 30′. Complete quadrantal triangle PDZ; then in triangle PDZ are given, angle PZD=52° 30′ (complement of amplitude), pol. dist. PD =74° 48′, and ZD=90°, to find colat. PZ, and thence the latitude.

By Rule XIV. $\cos.\ PD = \underline{\sin.\ PZ}\,.\,\cos.\ PZD$
whence PZ may be found.

Or simplifying formula . . sin. decl.=cos. lat. . sin. ampl.
∴ cos. lat.=sin. decl. . cosec. ampl.

Prob. 127.

Construct the following parts of fig. p. 125, horizon, celestial meridian, prime vertical; suppose D_2 to be the place where the sun rose, and D the place of setting; join PD and ZD; then in quadrantal triangle PZD there are given, colat. PZ, pol. dist. PD, and ZD=90°, to find angle PZD (the complement of amplitude), and hour-angle ZPD (the half the length of day), and thence the lengths of the day and night.

By Rule XIV. $\cos.\ PD = \sin.\ PZ\,.\,\underline{\cos.\ PZD}$
$\underline{\cos.}\ P = -\cot.\ PZ\,.\,\cot.\ PD$

placing the proper signs over the terms (p. 30), we shall find cos. PZD is positive, and cos. P negative, whence PZD=59° 55′ 30″, and ampl.=W. 30° 4′ 30″ N.

P=7h. 36m. 41s.=angle PZD_2

therefore the sun rises at 4h. 23m. 19s. A.M., sets at 7h. 36m. 41s. P.M., length of day 15h. 13m. 22s., length of night 8h. 46m. 38s., and amplitude at rising and setting 30° 4′ 30″ N.

Formulæ simplified . . sin. ampl.=sin. decl. . sec. lat.
cos. hour-angle=−tan. lat. . tan. decl.

PROB. 128.

Since the day is more than 12 hours long, the sun must rise (in north lat.) to the north of equator, as at S in fig. 97. Construct fig.; then in quadrantal triangle ZPS are given, colat. PZ=39° 12′, hour-angle P=7h., and ZS=90°, to find PZS, the complement of amplitude.

Rule XIV. cos. PZ=−cot. ZPS . $\overline{\text{tan. PZS}}$

whence PZS=N. 70° 55′ 30″ E.

and ampl.=E. 19° 4′ 30″ N.

Simplified . . cot. ampl.=−sin. lat. . tan. hour-angle.

PROB. 129.

Describe horizon, celestial meridian, prime vertical, and WQE, the celestial equator, fig. 71. Take some point X on the equator to represent the sun's place, and draw circle of altitude ZO, and circle of decl. PX; then in quadrantal triangle PXZ are given, zenith distance ZX=67° 4′, hour-angle P=3h., and PX=90°, to find colat. PZ, and thence the latitude.

By Rule XIV. cos. ZX=sin. PZ . cos. P

whence PZ=33° 26′ 30″, and lat.=56° 33′ 30″

Simplified . . cos. lat.=sin. alt. . sec. hour-angle.

PROB. 130.

Let X, fig. 74, be the place of the heavenly body; then in triangle PXZ are given, zen. dist. ZX, pol. dist. PX, and azimuth PZX, to find colat. PZ, and thence the latitude. The quantities given are not sufficient to enable us to find PZ by the common rules of spherical trigonometry. In cases of this kind it is usual to drop a perpendicular upon one of the sides, or side produced, thus forming two right-angled triangles, and then to apply, if possible, Rule XIII. Problems may often be solved by means of analytical trigonometry, when the common rules are insufficient, or tiresome if applied. We will give a few solutions of this problem, as they will be useful exercises in analytical trigonometry.

(1st solution.) By dropping a perpendicular, as XM, on the meridian.

(1.) In right-angled triangle XZM are given XZ and XZM, to find XM and ZM.

$\underline{\text{sin. XM}}=\text{sin. ZX . sin. XZM}\ \therefore\ \text{XM}=38^\circ\ 34'\ 30''$

$\text{cos. XZM}=\text{cot. XZ . }\underline{\text{tan. ZM}}\ \therefore\ \text{ZM}=30\ \ 12\ \ 45$

(2.) In right-angled triangle XPM are given XP and XM, to find PM.

$\text{cos. PX}=\underline{\text{cos. PM}}\text{ . cos. XM}\qquad \therefore\ \text{PM}=61^\circ\ \ 8'\ 45''$

whence $\text{PZ}=\text{PM}-\text{ZM}=30^\circ\ 56'$, and lat. $=59^\circ\ 4'$ N.

(2d solution.) By means of formula (U), *Trig.* Part II. p. 62.

(1.) Find angle P, by Rule X. Part I. p. 68, thus:

$$\text{sin. P : sin. PZX :: sin. XZ : sin. PX}$$
$$\therefore\ \text{P}=42^\circ\ 19'\ 30''$$

(2.) By (U).

$$\text{tan. }\tfrac{1}{2}\text{PZ}=\frac{\text{tan. }\frac{1}{2}(\text{PX}-\text{ZX})\text{ . sin. }\frac{1}{2}(\text{PZX}+\text{P})}{\text{sin. }\frac{1}{2}(\text{PZX}-\text{P})}$$

$$\therefore\ \tfrac{1}{2}\text{PZ}=15^\circ\ 27'\ 45'',\text{ and lat.}=59^\circ\ 4'\ 30''\text{ N.}$$

(3d solution.) By means of a subsidiary angle. *Trig.* Part II. p. 62 (L).

$$\text{cos. PZX}=\frac{\text{cos. PX}-\text{cos. PZ . cos. ZX}}{\text{sin. PZ . sin. ZX}}$$

If a=alt.
l=lat.
d=decl.

$$=\frac{\text{sin. }d-\text{sin. }l\text{ . sin. }a}{\text{cos. }l\text{ . cos. }a}$$

$$\therefore \text{sin. }d=\text{sin. }a\text{ . sin. }l+\text{cos. }l\text{ . cos. }a\text{ . cos. PZX}$$

$$=\text{sin. }a\ (\text{sin. }l+\text{cos. }l\text{ . cot. }a\text{ . cos. PZX})$$

Assume cot. a . cos. PZX=cot. θ . . . . (1)

$$\therefore \text{sin. }d=\text{sin. }a\ (\text{sin. }l+\text{cos. }l\text{ . cot. }\theta)$$

$$=\frac{\text{sin. }a\ (\text{sin. }l\text{ . sin. }\theta+\text{cos. }l\text{ . cos. }\theta)}{\text{sin. }\theta}$$

$$=\frac{\text{sin. }a\text{ . cos. }(l\backsim\theta)}{\text{sin. }\theta}$$

whence cos. $(l\backsim\theta)$=sin. d . sin. θ . cosec. a . . (2)

θ is found by (1)=120° 12′ 45″, and thence, by (2) $l\sim\theta$=61° 8′ 30″, and therefore l=59° 4′ 30″.

Computation.

(1). To find θ.
$-\qquad+\qquad-$
cot. θ=cot. a . cos. PZX
cot. a.... 0·037948
cot. PZX 9·727228
cot. θ. . . . 9·765176
$\therefore$ θ=180°−59° 47′ 15″

(2). To find $l\sim\theta$.
cosec. a .. 0·170317
sin. d......9·576689
sin. θ......9·936597
cos. $(l\backsim\theta)$ 9·683603
$\therefore$ $l\sim\theta$=61° 8′ 30″
$\therefore$ l=59 4 30″

(4th solution.) By elimination.

Let PX=p, ZX=a_1, XZM=Z,
PM=x, ZM=y, XM=z;
then colat.=$x-y$.

(1.) To find y. In triangle XZM .. cos. Z=tan. y. cot. a_1

$\therefore$ tan. y=cos. Z . tan. a_1=cos. az . cot. alt. (1)

(2.) To find x.

In triangle PXM .. cos. p =cos. x . cos. z

In triangle XZM .. cos. a_1=cos. z . cos. y

eliminate cos. z by division,

$$\frac{\cos.\ p}{\cos.\ a_1}=\frac{\cos.\ x}{\cos.\ y} \quad \text{or } \cos.\ x=\cos.\ p\ .\ \cos.\ y\ .\ \sec.\ a_1$$

$$=\sin.\ \text{decl.}\ .\ \text{cosec. alt.}\ .\ \cos.\ y \qquad (2)$$

hence by (1) $y=30°\ 12'\ 45''$; by (2) $x=61°\ 8'\ 30''$

$\therefore\ x-y$ the colat. $=30°\ 55'\ 45''$, and lat. $=59°\ 4'\ 15''$.

PROB. 131 (Fig. 74).

In this important problem are given the polar dist. $PX=90°-22°\ 30'$, the zen. dist. $ZX=90°-37°\ 20'$, and hour-angle $P=$ 2h. 15m., to find the latitude $=90°-PZ$.

As in the last problem, the common rules of trigonometry will not enable us to find the side PZ from the three known parts of the triangle. We may proceed as follows :

(1st solution.) By dropping a perpendicular XM.

(1.) In right-angled triangle PXM are given PX and P, to find PM. (Rule XIII. p. 72.)

$$\cos.\ P=\underline{\tan.\ PM}\ .\ \cot.\ PX$$

$$\therefore\ PM=63°\ 31'\ 15''.$$

Let h=hour-angle, d=declination, a=alt.

simplified . . . . $\underline{\tan.\ PM}=\cos.\ h\ .\ \cot.\ d$

(2.) In right-angled triangle PXM are given PX and P, to find MX.

$$\underline{\sin.\ MX}=\sin.\ P.\ \sin.\ PX.\ \text{(Rule XIII.)}$$

$$(\text{or } \underline{\sin.\ MX}=\sin.\ h\ .\ \cos.\ d) \quad \therefore\ MX=30°\ 53'.$$

(3.) In right-angled triangle ZXM are given ZX and XM, to find ZM.

$$\cos.\ ZX=\underline{\cos.\ ZM}\ .\ \cos.\ MX$$

$$(\text{or } \underline{\cos.\ ZM}=\sin.\ a\ .\ \sec.\ MX) \quad \therefore\ ZM=45°\ 2'\ 15''$$

$$\therefore\ PZ=PM-ZM=18°\ 29', \text{ and lat.}=71°\ 31'\ \text{N.}$$

(2nd solution.) By means of (U), *Trig.*, Part II. p. 62.

(1.) Find angle PZX. $\underline{\sin.\ PZX} : \sin P :: \sin.\ PX : \sin.\ ZX$

$$(\text{or } \sin.\ PZX=\cos.\ d\ .\ \sin.\ h\ .\ \sec.\ a)$$

$\therefore\ PZX=40°\ 12'\ 30''$, or its supplement $139°\ 47'\ 30''$, an ambiguous result; but it is evident by a rough construction of figure that the latter value must be taken. See fig. 74.

(2.) By formula (U), find $\frac{1}{2}$PZ (as in last problem)= 9° 14′ 20″, whence lat.$=90°-\text{PZ}=71° 31' 20''$.

(3d solution.) By means of a subsidiary angle.

Since (Part II., p. 62)

$$\cos.\ \text{P}=\frac{\cos.\ \text{ZX}-\cos.\ \text{PX}\ .\ \cos.\ \text{PZ}}{\sin.\ \text{PX}\ .\ \sin.\ \text{PZ}}=\frac{\sin.\ a-\sin.\ d\ .\ \sin.\ l}{\cos.\ d\ .\ \cos.\ l}$$

$$\therefore \sin.\ d\ .\ \sin.\ l+\cos.\ d\ .\ \cos.\ l\ .\ \cos.\ \text{P}=\sin.\ a$$

$$\text{or } \sin.\ d\ (\sin.\ l+\cos.\ l\ .\ \cot.\ d\ .\ \cos.\ \text{P})=\sin.\ a$$

$$\text{Assume } \cot.\ \theta=\cot.\ d\ .\ \cos.\ \text{P} \ \ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

$$\therefore \sin.\ d\ (\sin.\ l+\cos.\ l\ .\ \cot.\ \theta)=\sin.\ a$$

$$\text{or } \sin.\ d\ .\ \text{cosec.}\ \theta\ .\ \cos.\ (l\sim\theta)=\sin.\ a$$

$$\therefore \cos.\ (l\sim\theta)=\sin.\ a\ .\ \text{cosec.}\ d\ .\ \sin.\ \theta \ldots\ (2)$$

Formula (1) determines θ, and thence, by (2), $l\sim\theta$ may be found, and consequently the latitude l, as in the last problem.

(4th solution.) By elimination.

Let PX$=p$, ZX$=a_1$, P$=h$, PM$=x$,
ZM$=y$, XM$=z$; then colat.$=x\mp y$

(1.) To find x, or PM. In triangle XPM,

$$\cos.\ h=\cot.\ p\ .\ \tan.\ x \ \therefore \tan.\ x=\cos.\ h\ .\ \tan.\ p$$
$$=\cos.\ h\ .\ \cot.\ \text{decl.} \ldots (1)$$

(2.) To find y, or ZM. In triangle XPM,

$\cos.\ p=\cos.\ z\ .\ \cos.\ x$. In triangle ZXM, $\cos.\ a_1=\cos.\ y\ .\ \cos.\ z$

eliminate $\cos.\ z$ by division $\therefore \dfrac{\cos.\ p}{\cos.\ a_1}=\dfrac{\cos.\ x}{\cos.\ y}$

$$\therefore \cos.\ y=\cos.\ a_1\ .\ \cos.\ x\ .\ \sec.\ p$$
$$=\sin.\ \text{alt.}\ .\ \cos.\ \text{decl.}\ .\ \cos.\ x \ldots\ldots\ldots\ldots\ (2)$$

Formulæ (1) and (2) determine x and y, and thence the latitude, since colat.$=x\mp y$.

PROB. 132 (Fig. 76).

Ex. 1. Let X be the place of the sun in the ecliptic AXM, A the first point of Aries, and ARQ the celestial equator. Through X draw the circle of declination PXR; then in the spherical triangle AXR, right-angled at R, are given angle A the obliquity of the ecliptic, and AR the sun's right ascension, to find AX the longitude, and XR the declination.

To find longitude AX. Rule XIII.

$$\text{cos. A.} = \text{tan. AR} \;.\; \underline{\text{cot. AX}} \quad \therefore \text{AX} = 64^\circ\, 33'\, 15''$$

To find declination XR.

$$\text{sin. AR} = \underline{\text{tan. XR}} \;.\; \text{cot. A} \quad \therefore \text{XR} = 21^\circ\, 4'\, 15''\,\text{N.}$$

Ex. 2 (fig. 77). In this example the sun's right ascension exceeds 12 hours, it must therefore have passed the autumnal equinox, or first point of Libra, to the south side of the equator.

To find the place of the sun, let ALA_1 represent the whole of the celestial equator, A and A_1 the same point, namely, the first point of Aries, $ACLC_1A_1$ the ecliptic, cutting the equator in the two points A (or A_1) and L. Take AR equal to the sun's right ascension, and through R draw the circle of decl. RX; then X is the place of the sun, and its longitude is ACLX, and decl. RX.

To find longitude ACX, and decl. RX.

Subtract 12 hours from the right ascension ALR, the remainder is LR; and the angle RLX is also known, being the obliquity; then in triangle RLX,

$$\text{sin. RL} = \text{cot. obliq.} \;.\; \underline{\text{tan. decl.}} \quad \therefore \text{decl.} = 22^\circ\, 58'\,\text{S.}$$

$$\text{cos. obliq.} = \text{tan. LR} \;.\; \underline{\text{cot. LX}} \quad \therefore \text{LX} = 78^\circ\, 30'\, 15''$$

$$\therefore \text{long. ACLX} = 180^\circ + 78^\circ\, 30'\, 15'' = 258^\circ\, 30'\, 15''$$

Prob. 133.

Ex. 1 (fig. 78). Let X be the place of the heavenly body, AR the celestial equator, and AM the ecliptic. Through X draw PXR, a circle of decl., and P′XM, a circle of latitude, and join A and X by an arc of a great circle.

In the figure are given, AR the right ascension, XR the decl., and MAR the obliquity, to find AM the longitude, and XM the latitude.

(1.) In triangle AXR,

$\underline{\text{cos. }AX}=\text{cos. }RA\ .\ \text{cos. }XR\quad\therefore\ AX=48^\circ\ 45'\ 45''$

In same triangle,

$\text{sin. }AR=\text{tan. }XR\ .\ \underline{\text{cot. }XAR}\quad\therefore\ XAR=29^\circ\ 7'\ 0''$

$\therefore$ angle $XAM=29^\circ\ 7'\ 0''-23^\circ\ 27'\ 45''=5^\circ\ 39'\ 5''$

(2.) In triangle XAM,

$\text{cos. }XAM=\text{cot. }AX\ .\ \underline{\text{tan. }AM}\ \therefore$ long. $AM=48^\circ\ 37'\ 30''$

In same triangle,

$\underline{\text{sin. }XM}=\text{sin. }AX\ .\ \text{sin. }XAM\ \therefore$ lat. $XM=4^\circ\ 15'$ N.

Ex. 2. Since the right ascension exceeds 12 hours, the star must be to the east of the first point of Libra, as at X* in fig. 77. Draw XM a circle of lat., and XR′ a circle of decl., and join LX.

(1.) In right-angled triangle LXR′ are given, LR′= ∗R.A.—12h., decl. XR′=25° 51′ (N. decl.), to find LX and angle XLR′.

$\text{cos. }\underline{LX}=\text{cos. }LR'\ .\ \text{cos. }XR'\quad\therefore\ LX=66^\circ\ 19'\ 30''$

$\text{sin. }LR'=\underline{\text{cot. }XLR'}\ .\ \text{tan. }R'\quad\therefore\ XLR'=28^\circ\ 25'\ 45''$

(2.) In right-angled triangle XLM, given LX, and angle XLM(=XLR′+obliquity)=51° 53′ 30″, to find LM (=longitude—180°), and XM the latitude.

$\underline{\text{sin. }XM}=\text{sin. }LX\ .\ \text{sin. }XLM\quad\therefore\ XM=46^\circ\ 6'\ 15''$ N.

$\text{cos. }XLM=\text{cot. }LX\ .\ \underline{\text{tan. }LM}\quad\therefore\ LM=54^\circ\ 36'\ 30''$

whence long. $ACLM=180^\circ+54^\circ\ 36'\ 30''=234^\circ\ 36'\ 30''$.

Prob. 134 (Fig. 79).

Let A and B represent the two places in the same latitude, and PU, PV, meridians passing through them, and UV the arc of the equator intercepted between the meridians; then UV=difference of longitude between A and B=angle at P=10° 36′, and AP=PB=colatitude=45°.

In the spherical triangle APB are given two sides and included angle, to find the third side AB=7° 29′ 16″=449·25′, the distance between the two places in nautical miles.

---

* In some of the diagrams the R_1 should be marked R.

Or thus: since AP=PB, a perpendicular drawn from P upon AB will bisect the angle P and arc AB. Suppose in the figure an arc AD to be drawn perpendicular to AB, then in the right-angled triangle PDA are given AP=45° and angle APD=5° 18′, to find AD=half the distance AB.

Rule XIII......sin. AD=sin. AP . sin. APD.

Prob. 135 (Fig. 80).

Let UV be the arc of the equator intercepted between the meridians PB and PV passing through B and A, the two places; join AB. Then in the triangle BPA are given, PB=90°+34° 37′=124° 37′, PA=90°−50° 48′=39° 12′, and the included angle P=57° 18′, difference of longitude between the places.

Whence (by Rule IX. or X.) the third side AB may be found=99° 9′ 48″, or 5949·8 nautical miles.

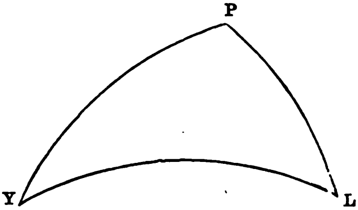
60 nautical miles=69·05 geographical miles:

hence to express the above distance in geographical miles we have 60 : 69·05 :: 5949·8 : x=6847·2 geographical or English miles.

Great Circle Sailing.

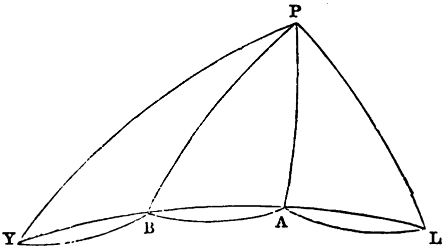
(1.) Find the shortest distance between Liverpool and New York.

lat. Liverpool ..	53° 25′ N.	long. Liverpool ..	2° 59′W.
„ New York .	40 42 N.	„ New York .	73 59 W.



Let PL, PY be the meridians of Liverpool and New York, and LY an arc of a great circle passing through the two places; then in the spherical triangle PYL are given PL=90°−53° 25′, PY=90°−40° 42′, and the included *angle P=71°*, the difference of longitude between the two places, to find (by Rule IX. or X.) the arc. $YL=47^\circ\,52'=2872$ nautical miles.

(2.) Take a point A on the arc YL, distant from Liverpool, L, 872 miles; take another point B, distant from A 1000 miles; then from B to New York, the distance $BY=1000$ miles. To find the latitude and longitude of each of the points A and B.



(1.) In the triangle PYL are given the three sides, to find (by Rule VIII.) the angle $L=75^\circ\,8'\,15''$.

(2.) To find PA, and thence the latitude of A.

In the triangle PLA are given, the sides AL ($=872'$ or $14^\circ\,32'$), PL, and the angle L, just found, to find PA (the colatitude of A)$=35^\circ\,21'$, and thence the lat. of $A=54^\circ\,39'$ N.

(3.) To find angle APL, and thence the longitude of A.

In the same triangle APL are given the three sides, to find the angle $APL=24^\circ\,47'\,15''$.

$$\begin{array}{rrrrl} \text{angle } APL= & 24^\circ & 47' & 15'' & \text{W.} \\ \text{long. of Liverpool}= & 2 & 59 & 0 & \text{W.} \\ \cline{2-4} \therefore \text{ long. of point } A= & 27 & 46 & 15 & \text{W.} \end{array}$$

(4.) To find PB, and thence the latitude of B.

In the triangle PBL are given the sides BL ($=1872'$ or $31^\circ\,12'$), PL, and the angle L, to find PB (the colat. of B) $=40^\circ$, and thence the lat. of $B=50^\circ$ N.

(5.) To find the angle BPL, and thence the longitude of B.

In the same triangle BPL are given the three sides, to find the angle $BPL=51^\circ\,10'$.

$$\begin{array}{rr} \text{angle BPL} = & 51^\circ\ 10'\ \text{W.} \\ \text{long. of Liverpool} = & \underline{\ \ 2\ \ \ 59\ }\ \text{W.} \\ \therefore\ \text{long. of point B} = & 54\ \ \ \ 9\ \ \text{W.} \end{array}$$

The points A and B being thus found on the line of shortest distance, the course and distance from Liverpool to A, thence from A to B, and thence from B to New York, may be found by the common rule of Navigation.

The course and distance from Liverpool to New York by Mercator's sailing are S. 75° 10′ 30″ W., 2982 miles, and the sum of the distances described by the ship sailing on the three loops, as above, is 2884, so that the distance saved by altering the course twice is about 98 miles.

The above method of finding the length of the arc of a great circle passing through two places on the surface of the earth, and of determining a certain number of convenient points thereon, to which the ship may be directed, so as to keep her as near to a great circle as possible, is direct and general. By dropping a perpendicular from P, or by means of special tables, the computation may be shortened, but only with the disadvantage of introducing a distinction of cases, and rendering the problem complicated. As the attempt at great circle sailing can only be of rare occurrence, it does not seem advisable to give formal rules for the purpose, nor to burden our books with more tables than are already in use.

By taking a greater number of points than A and B on the arc YL, so as to bring the ship oftener to the line of shortest distance, the sum of the distances sailed will approximate nearer to the shortest distance. It will be seen, however, that even on the above assumption of only altering the course once in about 1000 miles, the sum of the distances run exceeds by a very few miles the shortest distance, and that the absolute saving in distance between the two points Liverpool and New York is on this supposition nearly 100 miles.

Prob. 136 (Fig. 81).

Let AQ represent the celestial equator, A the first point of Aries, M the place of the moon, R the place of star; PM, PR circles of declination passing through M and R, and MR the arc of a great circle joining M and R; then MR is the distance required.

In the triangle MPR are given, PM=90°+decl. of moon =95° 19′ 0″, PR=90°−decl. of star=76° 49′ 45″, and P= diff. of right ascension=147° 46′ 0″, to find MR, the third side. (Rule IX. or X.)

Prob. 137 (Fig. resembling 69).

Construct the figure as follows: describe the horizon NWSE, celestial meridian NS, and prime vertical EW, as in preceding problems. On the meridian take NX=6° 10′, the altitude of the body; then X is the place of the body, and since its decl.=13° 10′, its polar dist. must be 76° 50′; therefore from X take a distance XP=76° 50′ towards the zenith; then P is the north pole, and PN the altitude of pole=latitude (*Trig.* Part I. p. 126).

Now PN=PX+XN=76° 50′+6° 10′=83° N.

Prob. 138 (Fig. 84).

Let AQ represent the celestial equator, AR the ecliptic, and S the place of the sun, M the place of the moon, P′ the pole of ecliptic, and P′M, P′S circles of latitude passing through M and S. Draw MS an arc of a great circle; then MS is the distance required.

In the quadrantal triangle P′MS are given, P′M=90°− moon's latitude=85° 5′ 30″, P′S=90°, and the angle P′ (the difference of longitude)=112° 58′ 45″, to find MS.

By Rule XIV., cos. MS=sin. P′M . cos. P′; putting over P′M and P′ their proper signs (see Part I. p. 29) we find cos. MS is negative, hence MS=112° 53′ 30″.

Prob. 139 (Fig. 85).

Let Z be the first position of the ship, and S the point arrived at after describing ZS, an arc of a great circle=100′, or 1° 40′. Through Z and S draw the meridians ZQ, SQ, meeting in pole P; then the angle SZQ is the initial course =45°, and the angle PSZ, or its opposite angle, is the course of the ship at S. To find this angle, we have in the triangle PZS the two sides (PZ colatitude, and ZS the distance run), and the included angle PZS=135°. Hence the angle at S may be found by Rule XI.

Prob. 140 (Figs. 86, 87).

In this problem it is not stated whether B is to the east or west of A. As the longitude of C depends on its position with respect to B as well as to A, there will be two solutions. (See figs. 86, 87).

Let A, B, C, be the three places; then we have given, PA (the colat. of A)=50°, PB (colat. of B)=40°, and AB=AC=BC=20°, to find CP, the colat. of C, and angle CPA, the difference of longitude between A and C, whence the longitude of C may be found.

(1.) In triangle ABP are given the three sides, to find the angle BAP=51° 48′.

(2.) In triangle BAC are given the three sides, to find the angle BAC=61° 1′ 30″; hence the angle PAC=9° 13′ 30″.

(3) In the triangle PAC are given, the two sides PA and AC, and included angle PAC, to find angle APC (the difference of longitude between A and C)=6° 13′ 30″, and the side PC (the colat.)=30° 23′ 15″.

If C is to the east of A (as in fig. 87),
then the long. of C=long. A+6° 13′ 30″=21° 13′ 30″ E.

If C is to the west of A (as in fig. 86),
then the long. of C=long. A−6° 13′ 30″=8° 46′ 0″ E.

Prob. 141 (Fig. 88).

Let T and R represent the places of the stars, C that of the comet, AEQ the ecliptic, and P the pole of ecliptic. Through the heavenly bodies T, C, R, draw the circles of latitude; then CE=lat. of comet, and AE the longitude (A being the first point of Aries).

(1.) In triangle TP_1R are given the side $P_1T=90°+$ lat. $=95° \ 28' \ 45''$, the side $P_1R=90°-$lat. $=89° \ 32' \ 30''$, and the included angle $TP_1R=$difference of longitudes$=80° \ 3' \ 15''$, to find the third side $TR=80° \ 8' \ 45''$.

(2.) In same triangle TP_1R are given the three sides, to find the angle $P_1TR=88° \ 34' \ 45''$.

(3.) In triangle RTC are given the three sides, to find angle $RTC=39° \ 32' \ 15''$.

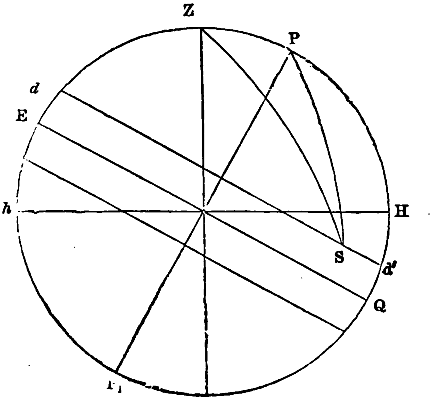
Hence the angle P_1TC is known, being the sum of the last two results$=128° \ 7'$.

(4.) In triangle P_1TC are given $TP_1=95° \ 28' \ 45''$, $TC=40° \ 12'$, and included angle $P_1TC=128° \ 7'$, to find $P_1C=118° \ 0' \ 15''$.

Hence CE the lat. $=28° \ 0' \ 15''$ S.

Lastly, in the same triangle P_1TC are given the three sides, to find the angle TP_1C (the difference of long. between comet and Aldebaran)$=35° \ 6' \ 40''$. Hence the longitude$=67° \ 12' \ 15''+35° \ 6' \ 40''=102° \ 18' \ 55''$.

PROB. 142.



Let the fig. be projected on the plane of the merid. PEP_1Q, Z the zenith of spectator, and Hh the plane of the horizon. Take $HP=54° \ 36'$, then P is the place of the elevated pole (*Trig.* Part I. p. 126); through P draw the diameter PP_1, and EQ at right angles to it. Then PP_1 is the axis of the heavens, and EQ the plane of the celestial equator. Draw *dd'* parallel to the equator, at the distance from it equal to the sun's declination 8° 30′ N., and let S be the sun's place when twilight commences; draw ZS, PS, arcs of great circles; then S is 18° below the horizon, and ZS=108°. In the triangle ZPS are given, ZS=108°, co-latitude ZP=35° 24′, and polar distance PS=81° 30′, to find the hour-angle ZPS=9h. 14m. 16s. Hence twilight begins at 2h. 45m. 44s. A.M., and ends at 9h. 14m. 16s. P.M. (the declination being supposed to remain unaltered).

PROB. 143 (Fig. 89).

(1st solution.) Let X and Y be the places of the sun at the times of the observations; PX, PY, the polar distances, each=66° 34′; and ZX, ZY, the zenith distances=50° 10′ and 61° 20′; and angle XPY=1h. 30m. (the interval between the observations). It is required to find PZ the colatitude, and thence the latitude. Draw XY an arc of a great circle.

(1.) In triangle PXY, given PX, PY, each 66° 34′, and included angle XPY=1h. 30m., to find XY=20° 37′ 21″.

(2.) In triangle PXY, given the three sides, to find angle PYX=85° 28′ 30″.

(3.) In triangle ZYX, given the three sides, to find angle ZYX=51° 41′.

(4.) Hence PYZ (=PYX−ZYX)=33° 47′ 30″.

Lastly. In triangle PYZ are given, PY, ZY, and included angle PYZ, to find the third side PZ=30° 42′ the co-lat. ∴ latitude=90°−30° 42′=59° 18′ N.

(2d solution [fig. 90]). Let X and Y be the places of the sun at the times of observation; PX, PY, the polar distances (which in this solution is always supposed to be equal). Bisect XY in M, and join PM, ZM; then PM is at right angles to XY, and therefore the angle PMZ is the complement of ZMX; draw ZE perpendicular to PM.

Let a_1 a_2 represent the altitudes at X and Y, h the half interval MPY between observations, d the decl., and l the *latitude.*

Let MX or $MY=x$, $PM=y$, $ZE=u$, $PE=v$, $\therefore ME=y-v$, $ZM=z$, and angle $ZMP=\theta$.

We have to compute the following arcs: (1) x; (2) y; (3) u; (4) $y-v$; then subtracting arc (4) from arc (2), v is known:* and lastly, knowing u and v in the triangle PZE, PZ the co-lat. can be found.

In triangle PMY...$\sin. x=\sin. h \,.\, \cos. d$ .... (1)

$\sin. d=\cos. y \,.\, \cos. x \quad \therefore \cos. y=\sin. d \,.\, \sec. x$ .... (2)

In triangle ZME...$\sin. u=\sin. \theta \,.\, \sin. z$ .... (α)

$$\cos. z=\cos. (y-v) \,.\, \cos. u$$

$\therefore \cos. (y-v)=\cos. z \,.\, \sec. u$ .... (β)

To eliminate $\sin. \theta$ and $\cos. z$ from (α) and (β), we have in the triangle ZMX and ZMY,

$$\cos. ZMX \text{ or } \sin. \theta=\frac{\sin. a_1-\cos. z \,.\, \cos. x}{\sin. z \,.\, \sin. x}$$

$$\cos. ZMY \text{ or } -\sin. \theta=\frac{\sin. a_2-\cos. z \,.\, \cos. x}{\sin. z \,.\, \sin. x}$$

adding $\quad 0=\sin. a_1+\sin. a_2-2 \cos. z \,.\, \cos. x$

$$\therefore 2 \cos. z \,.\, \cos. x=2 \sin. \tfrac{1}{2}(a_1+a_2) \,.\, \cos. \tfrac{1}{2}(a_1-a_2)$$

$$\text{or } \cos. z=\sin. \tfrac{1}{2}(a_1+a_2) \,.\, \cos. \tfrac{1}{2}(a_1-a_2) \,.\, \sec. x$$

Again, subtracting, $2 \sin. \theta=\dfrac{\sin. a_1-\sin. a_2}{\sin. z \,.\, \sin. x}$

$$\therefore \sin. \theta=\cos. \tfrac{1}{2}(a_1+a_2) \,.\, \sin. \tfrac{1}{2}(a_1-a_2) \,.\, \text{cosec. } z \,.\, \text{cosec. } x$$

Substituting these values of $\sin. \theta$ and $\cos. z$ in (α) and (β), we have,

$\sin. u=\cos. \tfrac{1}{2}(a_1+a_2) \,.\, \sin. \tfrac{1}{2}(a_1-a_2) \,.\, \text{cosec. } x$ .... (3)

$\cos. (y-v)=\sin. \tfrac{1}{2}(a_1+a_2) \,.\, \cos. \tfrac{1}{2}(a_1-a_2) \,.\, \sec. x \,.\, \sec. u$ (4)

---

* If the great circle drawn through X and Y pass, when produced, between P and Z, the perpendicular ZE will fall without the triangle PZM; in this case $ME=v-y$, and v is found by *adding* arcs (2) and (4) together. Or, since PE must evidently be less than 90°, and the colatitude PZ is always less than 90° (unless the latitude is nothing); consequently when MP+ME *is equal to or greater than* 90°, PE or v cannot be equal to MP+ME; hence v=the *difference* MP—ME; or *the difference* of arcs (2) and (4) must in such case be taken.

Lastly, in the right-angled triangle PZE,

cos. PZ or sin. l=cos. u . cos. v.

From this solution is deduced the rule in Navigation, Part I. p. 156, for finding the latitude by double altitude; observing that in finding y we must take the supplement of the arc given by the tables when the lat. and declination are of different names.

From solution 1 is derived the rule for finding the latitude by double altitude given in Navigation, Part I. p. 142. This method is perfectly general, since it may be applied to the same or different heavenly bodies, taken at the same or different times. The rule obtained from the second solution is considered more concise when the same heavenly body is observed, if its declination does not alter in the interval between the observations; as when two altitudes of a star are taken. It will also give a near approximate latitude from two altitudes of the sun; the declination in this case being supposed to remain invariable, and to be taken out of the Nautical Almanac for the middle time between the observations.

Prob. 144 (Fig. 91).

Proceeding in a similar manner as in solution 1 of last problem, we find XY=22° 42′; PYX=155° 38′; ZYX=89° 57′ ∴ PYZ=65° 41′; PZ=63° 36′ 53″, whence latitude=26° 23′ 7″ N.

Prob. 145 *a* (Fig. 92).

Let A and B be the two places in the same latitude, PA, PB, their meridians. Draw PD at right angles to AB, the great circle passing through the two places; then PD bisects the angle APB, since AP=BP; and D is the highest point reached by the ship sailing on a great circle from A to B. Produce also the meridians to the equator UV, and draw the parallel of latitude AD_1B.

(1.) In right-angled triangle APD, given PA=56° 9′, and *angle APD=half* difference of longitude between A and B=

68° 5′, to find AD=50° 23′ 45″ and PD=29° 6′ ∴ lat. of D=60° 54′, and AB=2 . AD=100° 47′ 45″, or 6047·5 nautical miles, the distance between A and B on a great circle.

(2.) To find the distance AD_1B on a parallel of latitude, we have (Trig. Part II. p. 80).

$$AD_1B = UV \,.\, \cos.\ AU = \text{diff. long.} \times \cos.\ \text{lat.}$$
$$= 8170 \cos.\ 33° \, 51' = 6785 \text{ miles.}$$

whence difference on the two circles=737·5 nautical miles.

Prob. 146 (Fig. 93).

Let A and B represent the two places, PA and PB their meridians. Draw the perpendicular PD, then the latitude of D is the highest attained by ship.

(1.) In triangle APB, given PA=56° 9′, PB=34° 2′, and angle APB=140° 27′, to find the angle PAB=20° 59′ 43″.

(2.) In right-angled triangle PAD, given PA=56° 9′ and angle PAB=20° 59′ 45″, to find PD=17° 18′ 45″ ∴ lat. of D=72° 41′ 15″.

Prob. 147 (Fig. 83).

Project this figure on the plane of the celestial meridian of the place. Let P be the pole, EQ the celestial equator, Z the zenith, S the place of the sun below the horizon H*h*, PS a circle of declination, and ZS a circle of altitude.

(1.) In triangle ZPS there are given, SP=90°+sun's decl.=100° 15′, PZ the colat.=39° 12′, and hour-angle P =7h., to find the third side SZ=107° 24′; whence 107° 24′ −90°=17° 24′ the sun's depression.

(2.) In the same triangle the three sides are now known, hence angle PZS, the azimuth, may be found=N. 84° 53′ W.

Prob. 148 (Fig. 94).

Let X and Y represent the two stars on the same circle of altitude. It is required to find the azimuth of X and Y, or the angle PZX.

(1.) In triangle PXY, given PX=73° 49′, PY=61° 35′ 34″, and included angle XPY=3h. 8m. 27s. (the difference of right ascensions), to find angle PXZ=65° 46′ 15″.

(2.) In triangle PXZ, given PX, PZ (the colat.), and angle PXZ, to find PZX=104° 55′; whence its supplement =S. 75° 5′ W. the bearing required.

Prob. 149 (Fig. 95).

Let M and X represent the two bodies on the same vertical circle ZM; it is required to find PZ, the colat. of the place, and thence the latitude.

(1.) In triangle PXM, given PX=77° 14′ 15″, PM=91° 42′ 30″, and included angle XPM=2h. 36m. 12s. (the difference of right ascensions), to find angle PMZ=68° 23′ 15″.

(2.) In triangle PMZ, given PM and ZM the moon's zen. dist., and included angle PMZ, to find colat. PZ=70° 4′ 22″, and therefore lat.=19° 55′ 38″ N.

Prob. 150 *a* (Fig. 72).

Suppose the heavenly body to rise at S and to pass the prime vertical at X, then PS=PX the star's polar distance; and EX=star's altitude when due east=20°. Let a=alt. at X; m=amplitude ES=11° 15′, x=lat. of spectator, and y=decl. of star.

Then $PS=PX=90°-y$, $PZ=90°-x$, and $ZX=90°-a$. In quadrantal triangle PZS (Rule XIV.)

$$\sin. y=\cos. x \,.\, \sin. m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

In right-angled triangle PZX (Rule XIII.)

$$\sin. y=\sin. x \,.\, \sin. a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$\therefore \text{ equating (1) and (2), } \sin. x \,.\, \sin. a=\cos. x \,.\, \sin. m$$

$$\text{or } \frac{\sin. x}{\cos. x}=\frac{\sin. m}{\sin. a} \;\therefore \tan. x=\sin. m \,.\, \text{cosec. } a$$

$$=\sin. 11° 15'. \text{ cosec. } 20° \;\therefore x=29° 42' \text{ N.}$$

Prob. 151 *a*.

Describe fig. to Prob. 107; take a distance EX=10° on the prime vertical, then X will be the place of the heavenly body when on the prime vertical, and m its place on the meridian. Let $XE=a$, and $mS=a_1$, the given altitudes when due east and due south; $l=$lat., and $d=$decl. Join PX; in right-angled triangle PZX,

$$\cos. PX=\cos. PZ . \cos. ZX, \text{ or } \sin. d=\sin l . \sin. a \ldots\ldots (1)$$

Now $l=ZQ=Zm+mQ=90°-a_1+d \therefore d=l+a_1-90°$

$$\therefore \sin. (l+a_1-90°)=\sin. l . \sin. a$$

$$\text{or } \sin. (l-\overline{90°-a_1})=\sin. l . \sin. a$$

$$\therefore \sin. l . \cos. (90°-a_1)-\cos. l . \sin. (90°-a_1)=\sin. l . \sin. a$$

$$\text{or } \sin. a_1-\cot. l . \cos. a_1=\sin. a$$

$$\therefore \cot. l . \cos. a_1=\sin. a_1-\sin. a$$

$$=2 \cos. \tfrac{1}{2}(a_1+a) . \sin. \tfrac{1}{2}(a_1-a)$$

and since $a_1=40°$, $a=10°$,

$\therefore \cot. l=2 \cos. 25°. \sin. 15°. \sec. 40°$ whence $l=58° 31'$ N.

PROB. 152 *a* (Fig. p. 125).

Let Y and V represent the places of the sun when due west and at six o'clock. $a=D_1V$, the alt. at 6 o'clock; $a_1=WY$, alt. when on the prime vertical;

$$\therefore ZV=90°-a, \text{ and } ZY=90°-a_1$$

Let $x=$lat., and $y=$decl., then $PZ=90°-x$, and $PV=PY=90°-y$. (By Rule XIII.)

In tri. ZPV, $\cos. (90°-a)=\cos. (90°-y) . \cos. (90°-x)$

$$\therefore \sin. a=\sin. y . \sin. x \ldots\ldots\ldots\ldots\ldots (1)$$

In tri. PZY, $\cos. (90°-y)=\cos. (90°-x) . \cos. (90°-a_1)$

$$\therefore \sin. y=\sin. x . \sin. a_1 \ldots\ldots\ldots\ldots (2)$$

Multiplying (1) and (2) together, and cancelling, we have,

$$\sin. a=\sin.^2 x . \sin. a_1, \text{ or } \sin.^2 x=\sin. a . \text{cosec. } a_1$$

which formula determines x; and dividing (1) by (2) we have,

$\sin.^2 y=\sin. a . \sin. a_1$ whence y may be found.

PROB. 153 *a* (Fig. p. 125).

Let V and D be the places of the body at 6 o'clock and when setting; join PD; then in the triangles PDZ and PVZ are given, D_1V the alt. at 6 o'clock, and DZP the complement of the amplitude, to find the latitude and decl.

Let a=alt. D_1V at 6 o'clock, m=amplitude WD, x= lat. $=90°-PZ$, y=decl.=complement of PV or PD.

Then in triangle PDZ . . $\sin. y=\cos. x . \sin. m$ . . . (1)
" " PVZ . . $\sin. a=\sin. y . \sin. x$ . . . (2)

Multiplying, and cancelling,

$$\sin. a=\sin. x . \cos. x . \sin. m=\tfrac{1}{2} \sin. 2x . \sin. m$$
$$\therefore \sin. 2x=2 \sin. a . \operatorname{cosec.} m$$

whence $2x=84°$, or $96°$ (the suppl.) and $x=42°$, or $48°$. Substituting these values of x in (1), we have $y=22° 20'$ or $20°$.

PROB. 154 *a* (Fig. p. 125).

Let V and D be the places of the body at 6 o'clock and when setting; join PD.

Let h=hour-angle DPZ when setting, a=alt. D_1V at 6 o'clock, x=lat. $=90°-PZ$, y=decl.=compl. of PD or PV.

Then (Rules XIV and XIII).

$$\cos. h=-\tan. y . \tan. x \quad \text{.......................} \quad (1)$$
$$\sin. a=\sin. y . \sin x \quad \text{.........................} \quad (2)$$

$$\text{From (1)} \cos. h=-\frac{\sin. y . \sin. x}{\cos. y . \cos. x}=-\frac{\sin. a}{\cos. y . \cos. x} \quad \text{(by 2)}$$

$$\therefore \cos. y . \cos. x . \cos. h=-\sin. a$$
$$\text{and } \cos. y . \cos. x=-\sin. a . \sec. h$$

But by (2) . . $\sin. y . \sin. x=\sin. a$

Adding and subtracting, we have

$$\cos. x . \cos. y+\sin. x . \sin. y=\sin. a . (1-\sec. h)$$
$$\cos. x . \cos. y-\sin. x . \sin. y=\sin. a . (1+\sec. h)$$
$$\text{or } \cos. (x-y)=\sin. a . (1-\sec. h) \quad \dots \quad (3)$$
$$\cos. (x+y)=-\sin. a . (1+\sec. h) \quad (4)$$

To reduce (3) and (4) to a logarithmic form.

$$\cos.(x-y)=\sin. a.\left(1-\frac{1}{\cos. h}\right)=\frac{\sin. a.(\cos. h-1)}{\cos. h}$$

$$=-2\sin. a.\sin.^2\frac{h}{2}.\sec. h \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

$$\cos.(x+y)=-\sin. a.\left(1+\frac{1}{\cos. h}\right)=-\frac{\sin. a.(\cos. h+1)}{\cos. h}$$

$$=-2\sin. a.\cos.^2\frac{h}{2}.\sec. h \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

From equations (5) and (6) the values of $x-y$ and $x+y$ may be determined, and thence x and y.

PROB. 155 *a* (Fig. p. 125).

Let h=hour-angle at setting, as at D, h_1=hour-angle at Y on prime vertical, x=lat., y=decl.

$$\text{Then } \cos. h=-\tan. x.\tan. y \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\cos. h_1=\cot. x.\tan. y \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$\text{Dividing (1) by (2), } \frac{\cos. h}{\cos. h_1}=-\frac{\tan. x}{\cot. x}=-\tan.^2 x$$

or $\tan.^2 x=-\cos. h.\sec. h$; which equation determines the value of x; and multiplying (1) and (2) together, we have $\tan.^2 y=-\cos. h.\cos. h_1$, whence the declination can be found.

PROB. 156 *a* (Fig. p. 125).

Let h=hour-angle at setting, h_1=hour-angle when west, t=interval $\therefore t=h-h_1$, d=decl., x=latitude.

$$\cos. t=\cos.(h-h_1)=\cos. h.\cos. h_1+\sin. h.\sin. h_1 \ldots\ldots \quad (1)$$

$$\text{By Rule XIV} \ldots \cos. h=-\tan. x.\tan. d \ldots\ldots\ldots\ldots \quad (2)$$

$$\ldots \quad \ldots \quad \text{XIII} \ldots \cos. h_1=\cot. x.\tan. d \ldots\ldots\ldots\ldots \quad (3)$$

$$\text{Multiplying (2) and (3)} \ldots \cos. h.\cos. h_1=-\tan.^2 d$$

$$\text{or } 2\cos. h.\cos. h_1=-2\tan.^2 d$$

Subtracting equation (1) from this equation,

$$\cos. h.\cos. h_1-\sin. h.\sin. h_1=-\cos. t-2\tan.^2 d$$

$$\therefore -\cos.(h+h_1)=\cos. t+2\tan.^2 d$$

This equation determines the value of $h+h_1$, the sum of the hour-angles; having this, and the difference, the angles h and h_1, and thence the latitude x, may be found. The operation is as follows:

To find value of $-\cos.\ (h+h_1)=\cos.\ t+2\ \tan.^2\ d$.

$$\text{tab. log. cos. } t=9\cdot864304 \quad \therefore \text{ log. cos. } t=\bar{1}\cdot864304$$

$$\therefore \cos.\ t=\cdot73165.$$

To find value of $2\ \tan.^2\ d$.

$$\log.\ (2\ \tan.^2\ d)=\log.\ 2+2\ \log.\ \tan.\ d-20=\bar{1}\cdot423162$$

$$\therefore 2\ \tan.^2\ d=\cdot26495$$

$$-\cos.\ (h+h_1)=\cdot73165+\cdot26495=\cdot9966$$

angle corresponding to log. of this in tables $=0^h\ 18^m\ 54^s$.

Now, since the cos. $(h+h_1)$ is *negative*, the value of $h+h_1$ must be either $12^h-18^m\ 54^s$ or $12^h+18^m\ 54^s$.

$$\begin{array}{rrrrcrrr}
\text{that is } h+h_1= & 11^h & 41^m & 6^s & \text{or} & 12^h & 18^m & 54^s \\
\text{and since } h-h_1= & 2 & 51 & 54 & ,, & 2 & 51 & 54 \\
\hline
\therefore 2h= & 14 & 33 & 0 & \text{or} & 15 & 10 & 48 \\
\text{and } h= & 7 & 16 & 30 & ,, & 7 & 35 & 24
\end{array}$$

With the first value of h the lat. $=42°$

............other.................... $=48$ nearly.

PROB. 157*a* (Fig. p. 125).

Let Z=angle PZV, the azimuth at 6 o'clock; m=amplitude WD, d=decl., l=lat.; then

Rule XIV..........$\sin.\ d=\cos.\ l\,.\,\sin.\ m$

......XIII.......... $\cos.\ l=\cot.\ d\,.\,\cot.\ Z$

Multiplying, $\sin.\ d\,.\,\cos.\ l=\cos.\ l\,.\,\sin.\ m\,.\,\cot.\ d\,.\,\cot.\ Z$

$$\therefore \sin.^2\ d=\sin.\ m\,.\,\cot.\ Z\,.\,\cos.\ d$$

$$\text{or } 1-\cos.^2\ d=\sin.\ m\,.\,\cot.\ Z\,.\,\cos.\ d$$

$$\text{whence } \cos.^2\ d+\sin.\ m\,.\,\cot.\ Z\,.\,\cos.\ d=1$$

Completing square, and extracting the root,

$$\cos.\ d=-\tfrac{1}{2}\sin.\ m\,.\,\cot.\ Z+\tfrac{1}{2}\sqrt{\sin.^2 m\,.\,\cot.^2 Z+4}$$

$$=-0\cdot06224+1\cdot00193=\cdot93969$$

$$\therefore \text{tab. log. cos. } d=9\cdot972984 \quad \therefore d=20°$$

and thence l may be found.

PROB. 158 *a* (Fig. p. 125),

Let a=altitude at 6 o'clock, m=meridian altitude, d and l the decl. and lat.

Then since lat. (QZ)=mer. zen. dist. (Zm)+decl. (mQ) (see construction of fig. to Prob. 107)

$$\text{or } l=90^\circ-m+d \quad \therefore\ d=l+m-90^\circ$$

$$\therefore\ \text{sin. } d=\text{sin. }(l+m-90^\circ)=-\text{sin. }(90^\circ-\overline{l+m})$$

$$=-\text{cos. }(l+m)$$

$$\text{But sin. } a=\text{sin. } l\,.\,\text{sin. } d=-\text{sin. } l\,.\,\text{cos. }(l+m)$$

$$\text{or } 2\text{ sin. } l\,.\,\text{cos. }(l+m)=-2\text{ sin. } a$$

$$\therefore\ \text{sin. }(2l+m)-\text{sin. } m=-2\text{ sin. } a$$

$$\text{or sin. }(2l+m)=\text{sin. } m-2\text{ sin. } a.$$

Calculation.

log. sin. m......$\bar{1}$·945935 ∴ nat. sine......0·88295

„ sin. a......$\bar{1}$·405141 ∴ nat. sine......0·25418

∴ sin. $m-2$ sin. a=0·37459=nat. sin. $(2l+m)$

∴ tab. log. sin. $(2l+m)$=9·573556

∴ $2l+m=158^\circ$ (suppl. of 22°)

whence $l=48^\circ$ N.

PROB. 159 *a* (Fig. p. 125).

Let a=meridian alt., h=hour-angle at rising or setting, d=decl., l=latitude.

$$\therefore\ d=l-\text{mer. zen. dist.}=l-\overline{90^\circ-a}=l+a-90^\circ.$$

In triangle PZD . . cos. $h=-$tan. d . tan. l

$$=-\text{tan. }(l+a-90^\circ)\,.\,\text{tan. } l=-\text{tan.}-(90^\circ-\overline{l+a})\,.\,\text{tan. } l$$

$$=\text{tan. }(90^\circ-\overline{l+a})\,.\,\text{tan. } l=\text{cot. }(l+a)\,.\,\text{tan. } l$$

$$=\frac{\text{cos. }(l+a)\,.\,\text{sin. } l}{\text{sin. }(l+a)\,.\,\text{cos. } l}$$

$$\therefore\ \frac{\text{cos. } h+1}{\text{cos. } h-1}=\frac{\text{cos. }(l+a)\,.\,\text{sin } l+\text{sin. }(l+a)\,.\,\text{cos. } l}{\text{cos. }(l+a)\,.\,\text{sin. } l-\text{sin. }(l+a)\,.\,\text{cos. } l}$$

$$\text{or}\ \frac{2\text{ cos.}^2\frac{h}{2}}{-2\text{sin.}^2\frac{h}{2}}=\frac{\text{sin. }(2l+a)}{-\text{sin } a.} \qquad \therefore\ \text{cot.}^2\frac{h}{2}\,.\,\text{sin. } a=\text{sin. }(2l+a)$$

whence $2l+a=29^\circ\ 13'$, or $150^\circ\ 47'$ (its supplement). Taking this last value, $2l=94^\circ\ 47'$ $\therefore\ l=47^\circ\ 23'\ 30''$

To avoid the ambiguity arising from obtaining the value of $2l+a$ in terms of the sine, let $z=90^\circ-a$ the zenith dist., then $d=l-z$

$$\therefore \cos.\ h=-\tan.\ d\ .\ \tan.\ l=-\tan.\ (l-z)\ .\ \tan.\ l$$

$$=\frac{-2\ \sin.\ (l-z)\ .\ \sin.\ l}{2\ \cos.\ (l-z)\ .\ \cos.\ l}=\frac{\cos.\ (2l-z)-\cos.\ z}{\cos.\ (2l-z)+\cos.\ z}$$

$$\therefore \frac{\cos.\ h+1}{\cos.\ h-1}=\frac{\cos.\ (2l-z)}{-\cos.\ z}\ \text{ or } \cos.\ (2l-z)=\cos.\ z\ .\ \cot.^2\frac{h}{2}$$

from which equation the value of l may be obtained free from ambiguity.

PROB. 160 a (Fig. p. 125).

Let h=hour-angle at setting, $h_{,}$=hour-angle when west.

By Rule XIV...... $\cos.\ h=-\tan.\ l\ .\ \tan.\ d$

.... XIII...... $\cos.\ h_{,}=\cot.\ l\ .\ \tan.\ d$

adding, $\cos.\ h+\cos.\ h_{,}=(\cot.\ l-\tan.\ l)\ .\ \tan.\ d$

subtracting, $\cos.\ h-\cos.\ h_{,}=-(\tan.\ l+\cot.\ l)\ .\ \tan.\ d.$

$$\text{dividing,}\quad \frac{\cos.\ h+\cos.\ h_{,}}{\cos.\ h-\cos.\ h_{,}}=-\frac{\cot.\ l-\tan.\ l}{\tan.\ l+\cot.\ l}$$

$$\text{or}\quad \frac{\cos.\ h+\cos.\ h_{,}}{-(\cos.\ h-\cos.\ h_{,})}=\frac{\cot.\ l-\tan.\ l}{\tan.\ l+\cot.\ l}$$

$$\therefore \frac{2\ \cos.\ \tfrac{1}{2}(h+h_{,})\ .\ \cos.\ \tfrac{1}{2}(h-h_{,})}{2\ \sin.\ \tfrac{1}{2}(h+h_{,})\ .\ \sin.\ \tfrac{1}{2}(h-h_{,})}=\frac{1-\tan.^2 l}{\tan.^2 l+1}=\cos.\ 2l$$

$$\text{or } \cot.\ \tfrac{1}{2}(h+h_{,})\ .\ \cot.\ \tfrac{1}{2}(h-h_{,})=\cos.\ 2l$$

$$\therefore \cot.\ \tfrac{1}{2}(h+h_{,})=\cos.\ 2l\ .\ \tan.\ \tfrac{1}{2}(h-h_{,})$$

which equation determines $\frac{1}{2}(h+h_{,})$, and with $\frac{1}{2}(h-h_{,})$ already known, the angles h and $h_{,}$ may be found, and thence by Rule XIII. we can find d.

PROB. 161 a (Fig. 96).

Suppose the heavenly body to rise at S, and to describe the parallel of declination SX. Let a circle of altitude ZO *touch the* parallel SX at X; then X is the point where the azimuth is the greatest; the azimuth increasing from the point S, where the heavenly body rose, to X, and beginning to decrease after passing X. The angle PXZ is evidently a right angle; whence the required azimuth, hour, and altitude may all be found from the right-angled triangle PZX.

(1.) To find greatest azimuth PZX.

$\cos. d = \cos. l . \sin. \text{az.} \qquad \therefore \sin. \text{az.} = \cos. d . \sec. l$

(2.) To find the hour-angle ZPX, or h.

$$\cos. h. = \cot. d . \tan. l.$$

(3.) To find altitude XO or a.

$\sin. l = \sin. d . \sin. a \qquad \therefore \sin. a = \sin. l . \text{cosec.} d$

PROB. 162 *a* (Fig. 96).

The preceding problem furnishes an explanation of the fact, that when the sun's declination is greater than the latitude of a place, and of the same name with it, the shadow of an upright object on a horizontal plane goes backward each day during a certain period, which may be computed, as well as the arc through which it goes back.

At the point X the sun appears stationary in azimuth. PZS is its bearing when rising, PZO its bearing at point X. The angle SZO or arc SO will therefore denote the number of degrees the bearing has increased from sun rising, or that the shadow of the object has gone back; and the period during which the retrograde motion has taken place will be measured by the angle SPX.

(1.) To find bearing PZS at rising.

In triangle PZS, $\sin. d = \cos. l . \cos. \text{PZS}$

$$\begin{array}{lr} \text{whence} & \text{PZS} = 64^\circ\ 55'\ 30'' \\ \text{By last prob.} & \text{PZX} = 77\ \ 28\ \ \ 0 \\ \hline \therefore \text{shadow goes back} & = 12\ \ 32\ \ 30 \end{array}$$

(2.) To find hour-angle ZPS at rising.

In triangle PZS, $\overset{-}{\cos.\ \mathrm{ZPS}} = -\overset{+}{\tan.\ l} \ .\ \overset{+}{\tan.\ d}$

$$\therefore \mathrm{ZPS} = 6\text{h. } 36\text{m. } 22\text{s.}$$

by last prob. $\mathrm{ZPX} = 2 \quad 12 \quad 7$

$\therefore$ time of shadow's going back $= 4 \quad 24 \quad 15$

PROB. 163 *a* (Fig. 97).

(1st solution.) Let S and M be the respective places of the sun and moon at the time of observation; d and $-d$ the decl. of sun and moon respectively; that of the moon being taken negatively, since the bodies are on different sides of the equator; h and $h_{,}$ their hour-angles.

Then $h - h_{,} = 1\text{h. } 53\text{m. } 42\text{s.}$

In triangle ZPS, $\cos.\ h = -\tan.\ l\ .\ \tan.\ d$ .... (1)

In triangle ZPM, $\cos.\ h_{,} = -\tan.\ l\ .\ \tan.\ (-d_{,})$

$= \tan.\ l\ .\ \tan.\ d_{,}$ .... (2)

dividing (2) by (1), $\dfrac{\cos.\ h_{,}}{\cos.\ h} = \dfrac{\tan.\ d_{,}}{-\tan.\ d}$

or $\dfrac{\cos.\ h_{,} + \cos.\ h}{\cos.\ h_{,} - \cos.\ h} = \dfrac{\tan.\ d_{,} - \tan.\ d}{\tan.\ d_{,} + \tan.\ d}$

$$\therefore \frac{2\cos.\tfrac{1}{2}(h_{,}+h)\ .\ \cos.\tfrac{1}{2}(h_{,}-h)}{-2\sin.\tfrac{1}{2}(h_{,}+h).\sin.\tfrac{1}{2}(h_{,}-h)} = \frac{\sin.\ d_{,}.\cos.d - \cos.\ d_{,}.\ \sin.\ d}{\sin.\ d_{,}\ \cos.d - \cos.\ d_{,}.\ \sin.\ d}$$

or $-\cot.\ \tfrac{1}{2}(h_{,}+h)\ .\ \cot.\ \tfrac{1}{2}(h_{,}-h) = \dfrac{\sin.\ (d_{,}-d)}{\sin.\ (d_{,}+d)}$

$$\therefore \cot.\ \tfrac{1}{2}(h_{,}+h) = -\sin.\ (d_{,}-d)\ .\ \mathrm{cosec.}\ (d_{,}+d)\ .\ \tan.\tfrac{1}{2}(h_{,}-h)$$
$$= \sin.\ (d_{,}-d)\ .\ \mathrm{cosec.}\ (d_{,}+d)\ .\tan.\tfrac{1}{2}(h-h_{,})$$

whence $\tfrac{1}{2}(h_{,}+h) = 5\text{h. } 53\text{m. } 29\text{s.}$

and since $\tfrac{1}{2}(h-h_{,}) = 0 \quad 56 \quad 51$

$\therefore h = 6 \quad 50 \quad 20$

hence app. time=12h.—6h. 50m. 20s.=5h. 9m. 40s. A.M.

To find lat. l. $\overset{-}{\cos.\ h} = -\overset{+}{\tan.\ l}\ .\ \overset{+}{\tan.\ d}$

or $\tan.\ l = \cos.\ h\ .\ \cot.\ d \quad \therefore l = 50^\circ\ 18'\ 20''$ N.

(2d solution.) More easily, thus:

(1.) In triangle PMS, given PS, PM, and included angle SPM, to find angle PSM=128° 33′ 30″.

(2.) In right-angled triangle NPS, given PS and angle PSN (supplement of PSM), to find lat. NP=50° 18′ 20″ N.

PROB. 164 *a*.

Let h and $h_{,}$ denote the hour-angles of the sun at rising or setting at the two places: then since it rises and sets at the same instant of absolute time at the two places, the difference of the hour-angles h and $h_{,}$ or $h-h_{,}$ must be equal to the difference of their reckoning of time, or to their difference of long. (in time)=28° 46′.

Let l and $-l_{,}$ denote the latitudes of Dublin and Pernambuco: the latitudes being affected with different signs, since the places are on opposite sides of the equator.

$$\text{Then } \cos. h = -\tan. l \,.\, \tan. d \ldots\ldots\ldots\ldots\ldots \text{(1)}$$

$$\cos. h_{,} = -\tan. (-l_{,}) \,.\, \tan. d = \tan. l_{,} \,.\, \tan. d \ldots\ldots \text{(2)}$$

$$\therefore \frac{\cos. h}{\cos. h_{,}} = \frac{\tan. l}{-\tan. l_{,}}, \text{ or } \frac{\cos. h_{,} + \cos. h}{\cos. h_{,} - \cos. h} = \frac{\tan. l_{,} - \tan. l}{\tan. l_{,} + \tan. l}$$

$$\therefore \frac{\cos. \frac{1}{2}(h_{,}+h) \,.\, \cos. \frac{1}{2}(h_{,}-h)}{-\sin. \frac{1}{2}(h_{,}+h) . \sin. \frac{1}{2}(h_{,}-h)} = \frac{\sin. (l_{,}-l)}{\sin. (l_{,}+l)} = \frac{-\sin. (l-l_{,})}{\sin. (l+l_{,})}$$

$$\text{or } \cot. \tfrac{1}{2}(h_{,}+h) = -\sin. (l-l_{,}) \,.\, \text{cosec.} (l+l_{,}) \,.\, \tan. \tfrac{1}{2}(h-h_{,})$$

This equation determines half the sum of the hour-angles= 101° 40′ 30″, and thence, with $h-h_{,}$ already known, the values of h or $h_{,}$, and by (1) the declination d=18° 6′. Then the days on which the declination of the sun is the value of d thus found will be those required.

PROB. 165 *a*.

Let O be the center of a square ABCD, each of whose sides is 300 feet; draw obelisk OP perpendicular to the plane of ABCD, and make it equal to 60 feet; join AO and AP; then the angle PAO measures the altitude of the sun when the shadow AO of obelisk reaches to A the corner *of the square.*

To find the angle PAO.

Draw OE perpendicular to AD; then $AE=EO=\frac{1}{2}AD=$ 150 feet.

$$\therefore AO=AE\sqrt{2}=150\sqrt{2}.$$

$$\text{and tan. } PAO=\frac{OP}{AO}=\frac{60}{150\sqrt{2}}=\frac{6}{30}\sqrt{2}=\frac{\sqrt{2}}{5}=\cdot 2\sqrt{2}$$

whence angle PAO=15° 47′ 30″, the sun's altitude.

Prob. 166 *a* (Fig. 98).

Let ZQH represent the celestial meridian, H the horizon, Z the zenith, Q the equator, S′ and S the places of the sun on the longest and shortest days.

$$\text{Then } ZQ=\text{lat. } (x)\text{, and } QH=90^\circ-x$$
$$SQ=S'Q=\text{decl. } (d)=23^\circ\ 28'$$

Let NG (the shadow of object PN when sun is at S′)$=y$; then NC (its shadow when at S)$=7y$.

$$\text{Now angle NPC or arc } ZS=x+d$$
$$\text{and angle NPG or arc } ZS'=x-d$$

$$\text{In triangle NPC .. } 7y=PN\ .\ \tan.\ NPC=PN\ .\ \tan.(x+d)$$
$$\text{.. .. NPG .. } y=PN\ .\ \tan.\ NPG=PN\ .\ \tan.(x-d)$$

$$\text{dividing .. } 7=\frac{\tan.\ (x+d)}{\tan.\ (x-d)}=\frac{\sin.\ 2x+\sin.\ 2d}{\sin.\ 2x-\sin.\ 2d}$$

$$\therefore \frac{8}{6}\text{ or }\frac{4}{3}=\frac{\sin.\ 2x}{\sin.\ 2d}\quad\text{or }\sin.\ 2x=\frac{4}{3}\sin.\ 2d$$

$$\text{whence } x=38^\circ\ 27'\ 45''=\text{lat.}$$

Prob. 167 *a* (Fig. 98).

Let $NG=x$, $NC=y$, $l=$lat., $d=$decl.

then $x=PN\ .\ \tan.\ (l-d)$, $y=PN\ .\ \tan.\ (l+d)$

$$\therefore \frac{y}{x}=\frac{\tan.\ (l+d)}{\tan.\ (l-d)}=\frac{\sin.\ 2l+\sin.\ 2d}{\sin.\ 2l-\sin.\ 2d}$$

$$=\frac{\sin.\ 90^\circ+\sin.\ 30^\circ}{\sin.\ 90^\circ-\sin.\ 30^\circ}=\frac{1+\frac{1}{2}}{1-\frac{1}{2}}=\frac{3}{1}$$

$$\text{or } y:x::3:1.$$

Prob. 168 *a*.

In fig. 98, if S′ be the place of the sun, and NG the shadow of object PN, then

$$\frac{\mathrm{PN}}{\mathrm{NG}}=\frac{3}{5}=\text{tan. PGN}=\text{tan. sun's alt.}=\cdot 6$$

$$\therefore \text{log. tan. alt.}=9\cdot 778151 \qquad \therefore \text{alt.}=30^\circ\ 58'.$$

Then in triangle PZX (fig. 73), the three sides are given, viz. PZ=56° 30′, PX=79° 45′, and ZX=59° 2′, to compute the hour-angle ZPX, or h; hence h=3h. 58m. 4s.

Prob. 169 *a* (Fig. 99).

Let AB represent the shadow of perpendicular stick AD, and AC its shadow when the stick is placed in the position AT perpendicular to ray SC, in which position it will manifestly throw the greatest shadow. Now the distance of S is so great compared with BC that the angle BSC has no assignable magnitude, and the rays SB, SC may therefore be considered parallel, or the angle ABD=angle ACT=sun's altitude required.

$$\text{In triangle ABD .. AD}=\text{AB . tan. alt.}=4\text{ tan. alt.}$$

$$\text{In ATC .. AT}=\text{AD}=\text{AC . sin. alt.}=5\text{ sin. alt.}$$

$$\therefore 5\text{ sin. alt.}=4\text{ tan. alt.}=4\text{ sin. alt. . sec. alt.}$$

$$\therefore \text{sec. alt.}=\tfrac{5}{4}=1\cdot 25$$

$$\text{log. sec. alt.}=10\cdot 096910, \quad \therefore \text{alt.}=36^\circ\ 52'\ 15''.$$

Prob. 170 *a* (Fig. 100).

Let A be the observer, B the shadow of cloud, C and D the place of cloud and sun. Since the distance AB will subtend no sensible angle at the sun, the rays AD and BD may be considered parallel, and angle CBE=DAB=sun's alt.=22°, and the angle of elevation of cloud=20°. Therefore ACB=2°.

$$\text{In tri. ABC .. }\frac{\mathrm{BC}}{\mathrm{AB}}=\frac{\text{sin. }20^\circ}{\text{sin. }2^\circ} \quad \therefore \mathrm{BC}=\frac{\text{AB . sin. }20^\circ}{\text{sin. }2^\circ}$$

In right-angled triangle CBE .. CE=CB . sin. 22°

$\therefore$ CE=AB . sin. 20° . sin. 22° . cosec. 2°=1468.

Prob. 171 *a* (Fig. 101).

Let h denote the hour-angle of the sun at rising, or half the length of the day ; and since the meridian altitude is less than the latitude, the declination must be of a different name to latitude, or south.

In quadrantal triangle PZS, $\cos. h = -\tan. l . \cot.(90°+d)$ $=\tan. l . \tan. d = \tan. d$, since $\tan. l = \tan. 45° = 1$.

$$\therefore \cos. h = \tan. (z-l) \quad \text{if } z = \text{zen. dist.}$$

$$=\frac{\tan. z - \tan. l}{1+\tan. z . \tan. l} = \frac{\tan. z - 1}{\tan. z + 1}$$

$$\therefore \frac{\cos. h+1}{\cos. h-1} = -\tan. z, \text{ or } \frac{1+\cos. h}{1-\cos. h} = \tan. z$$

$$\therefore \cot.^2\frac{h}{2} = \tan. z = \cot. \text{alt.} = \cot. 30° = \sqrt{3}$$

$$\text{hence } \cot.\frac{h}{2} = \sqrt[4]{3} \text{ and } \tan.\frac{h}{2} = \frac{1}{\sqrt[4]{3}}$$

Prob. 172 *a*.

Let z=zenith dist. $\quad\therefore z=2l$

and (fig. 66) $z=l+d \quad \therefore 2l=l+d$ or $l=d$.

In triangle PZS (fig. 101), $\cos. h = -\tan. l . \cot. (90°+d)$

$$=\tan. l . \tan. d = \tan.^2 l$$

$$\therefore \tan.^2 l = \cos. 7\text{h., whence } l = 26° 58'.$$

Prob. 173 *a* (Fig. p. 125).

Suppose D_2 the place where the star rose ; draw PD_2 and ZD_2 ; let h=hour-angle, and z=meridian zenith distance, l=lat., and d=decl. Then

$\cos. h = -\tan. l . \tan. d = -\tan. d$ since $l=45°$

but $d=l-z \quad \therefore \cos. h = -\tan. (l-z)$

$$=-\frac{\tan. l - \tan. z}{1+\tan. l . \tan. z} = \frac{\tan. z - 1}{\tan. z + 1}$$

$$\text{or } \frac{\cos. h+1}{\cos. h-1}=-\tan. z=-\cot. \text{alt.}$$

$$\therefore \cot. \text{alt.}=\cot^2\frac{h}{2}=\cot.^2\, 60°=\frac{1}{3} \quad \therefore \tan. \text{alt.}=3.$$

PROB. 174 *a* (Fig. 102).

Let $S_{,}$ be the apparent place of the sun when its upper limb touches the horizon, S its apparent place when lower limb touches horizon; then angle $S_{,}PS$ measures the time elapsed between the upper and lower limbs being in horizon.

$$S_{,}PS=ZPS_{,}-ZPS.$$

(1) In triangle $ZPS_{,}$, given ZP=40°, $ZS_{,}$=90° 16′, and $PS_{,}$=84° 22′ $\therefore$ angle $ZPS_{,}$=6h. 28m. 41s.

(2) In triangle ZPS, given ZP=40°, ZS=90°−16′=89° 44′, and PS=84° 22′ $\therefore$ ZPS=6h. 25m. 22s.

$$\therefore \text{angle } S_{,}PS=3\text{m. }19\text{s.}$$

PROB. 175 *a* (Fig. 103).

Let AB represent the arc of the parallel of decl. described by sun's center, P the pole of the heavens; then considering the triangle APB as a spherical triangle (which may be done in this case, since arc AB being small does not differ much from the corresponding arc of a great circle), we have the three sides of APB given to find the angle P, the time of semidiameter passing meridian. Now (*Trig.* Part II. p. 62)

$$\sin.^2\frac{P}{2}=\text{cosec.AP.cosec.PB.sin.}\tfrac{1}{2}(AB+AP-PB).\text{sin.}\tfrac{1}{2}(AB-\overline{AP-PB}).$$

$$=\text{cosec.}^2\, AP\, .\, \sin.^2\, AB \quad \text{since } AP=BP$$

$$\therefore \sin.\frac{P}{2}=\sec.\, d\, .\, \sin.\, 8'\, 8{\cdot}6'' \text{ and } P=1\text{m. }11\text{s. nearly.}$$

Or thus: Let AB represent an arc of a small circle equal in length to the sun's semidiameter, A the sun's center. Through A and B draw circles of declination PU, PV, cutting the equator in U and V; then the time of the point A describing the arc AB will be measured by the arc UV, or the angle UPV.

Now (*Trig.* Part II. p. 80) $UV = AB \sec. AU = 16'\ 17{\cdot}3''. \sec. d$

$$\text{or arc } UV \text{ (in time)} = \frac{977{\cdot}3}{15} \sec. d = 1\text{m. } 10{\cdot}22\text{s.}$$

(this is independent of the sun's apparent motion from west to east).

Prob. 176 *a* (Fig. 104).

Let A be the first station of ship, B the second, and C the point of land.

(1) In triangle ABC, given AB=6 miles, angle CAB (the difference of bearing between C and B)=2 points, and angle ACB=$4\frac{1}{2}$ points, to find AC.

(5) Draw AD due east, and CD at right angles to it, then CD=difference of latitude between A and C, and (considering ADC as a right-angled plane triangle) $CD = AC . \sin. CAD = AC . \sin. 4 \text{ points} = 5'\ 15''$

$$\therefore \text{lat. of } C = 59^\circ\ 6' + 5'\ 15'' = 59^\circ{\cdot}11'\ 15''$$

also (the triangle ADC being isosceles) $AD = 5'\ 15''$

To find the difference of longitude between A and C, we have (*Nav.* p. 43) $AD = \text{diff. long.} . \cos. \text{lat. } A$, whence diff. long.$=10'\ 13''$ E.

$$\therefore \text{long. of } C = 6^\circ\ 15' + 10'\ 13'' = 6^\circ\ 25'\ 13'' \text{ E.}$$

Prob. 177 *a* (Fig. 97).

Let S and M be the places of sun rising on the longest and shortest days,

h =hour-angle of sun at S,

$h_{,}$= .... M;

then $2(h - h_{,})$=difference in length of day.

In triangle ZPS .. $\cos. h = -\tan. d . \tan. l$ .... (1)

............ZPM .. $\cos. h_{,} = +\tan. d . \tan. l$ .... (2)

$\therefore \cos. h = -\cos. h_{,} = \cos. (12\text{h.} - h_{,})$ $\therefore h = 12 - h_{,}$

or $h - h_{,} = 12 - 2h_{,} = \frac{1}{2}$ difference required.

Now in (2), $\cos. h_{,} = \tan. d . \tan. l = \tan. d$, since $l = 45^\circ$.

Calculation.

log. tan. d......9·637611 = log. cos. $h_{,}$

$\therefore h_{,}=$4h. 17m. 5s., and $2h_{,}=$8h. 34m. 10s.

$\therefore h-h_{,}=(12-2h_{,})=$3h. 25m. 50s.

and difference in length of days=6h. 51m. 40s.

Prob. 178 *a* (Fig. 97).

Let h=hour-angle of the sun at S,

$h_{,}$.............................. M $\therefore h=h_{,}+3$.

In triangle ZPS......cos. $(h_{,}+3)=-$tan. d. tan. l......... (1)

.... ZPM............ cos. $h_{,}=$tan. d. tan. l............ (2)

$\therefore$ cos. $(h_{,}+3)=-$cos. $h_{,}=$cos. $(12-h_{,})$

$\therefore h_{,}+3=12-h_{,}$ or $h_{,}=$4h. 30m.; whence by (2), $l=41° 24'$.

Prob. 179 *a*.

Let h and $h_{,}$ be the hour-angles; then $h-h_{,}=$1h.

d and $d_{,}$ the decl. at these times, viz. 20° and 10°

then cos. $h=-$tan. l. tan. d..................... (1)

cos. $h_{,}=-$tan. l. tan. $d_{,}$ .... (2)

$$\therefore \frac{\text{cos. } h}{\text{cos. } h_{,}}=\frac{\text{tan. } d}{\text{tan. } d_{,}} \text{ or } \frac{\text{cos. } h+\text{cos. } h_{,}}{\text{cos. } h-\text{cos. } h_{,}}=\frac{\text{tan. } d+\text{tan. } d_{,}}{\text{tan. } d-\text{tan. } d_{,}}$$

$$\text{or } \frac{2 \text{ cos. } \frac{1}{2}(h+h_{,}) . \text{ cos. } \frac{1}{2}(h-h_{,})}{-2 \text{ sin. } \frac{1}{2}(h+h_{,}) . \text{sin. } \frac{1}{2}(h-h_{,})}=\frac{\text{sin. } (d+d_{,})}{\text{sin. } (d-d_{,})}$$

$$\therefore -\text{cot. } \tfrac{1}{2}(h+h_{,}) . \text{ cot. } \tfrac{1}{2}(h-h_{,})=\frac{\text{sin. } 30°}{\text{sin. } 10°}$$

$$-\qquad\qquad\qquad +\qquad\qquad +\qquad\qquad +$$

$\therefore -$cot. $\frac{1}{2}(h+h_{,})=$tan. 7° 30′. sin. 30° . cosec. 10°

whence $\frac{1}{2}(h+h_{,})=$7h. 23m. 3s.

and $\frac{1}{2}(h-h_{,})=$0 30 0

$\therefore h=$7 53 3

with this value of h, the lat. (by 1)=52° 27′ N.

Prob. 180 *a* (Fig. 97).

Let h be the hour-angle of sun at S,

$h_{,}$.............................. M $\therefore h=3h_{,}$

$$\text{then } \cos.\ h = -\tan.\ l\ .\ \tan.\ d \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\cos.\ h_{,} = +\tan.\ l\ .\ \tan.\ d \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$\therefore \cos.\ h = -\cos.\ h_{,} = \cos.\ (12\text{h.} - h_{,})$$

$$\therefore h = 12 - h_{,} \quad \text{or } 3h_{,} = 12 - h_{,} \qquad \therefore 4h_{,} = 12$$

and $h_{,} = 3$ hours; whence, by (2), $l = 58° 27'$ N.

PROB. 181 *a*.

Let h and $h_{,}$ be the hour-angles, then $h_{,} = \frac{3}{5}h$

and $\cos.\ h = -\tan.\ d\ .\ \tan.\ l, \quad \cos.\ h_{,} = \tan.\ d\ .\ \tan.\ l$

$$\therefore \cos.\ h = -\cos.\ h_{,} = \cos.\ (12 - h_{,}) \ \therefore h = 12 - h_{,} = 12 - \frac{3}{5}h$$

$$\therefore \frac{8h}{5} = 12 \quad \text{and } h = \frac{60}{8} = 7\text{h. } 30\text{m.}$$

with this value of h the lat. $= 41° 24'$ N.

PROB. 182 (Fig. 105.)

Let A be the place sailed from, B the place arrived at; the arc $AA_{,}$ = arc $A_{,}B = a$, and $BC = b$; EQ the arc of equator between meridians of A and $A_{,}$.

Let x = lat. of A; then $x - a$ = lat. of B. *Trig.* Part II. p. 80,

$$\frac{CB}{EQ} = \frac{b}{EQ} = \cos.\ BQ = \cos.\ (x-a); \quad \frac{AA_{,}}{EQ} = \frac{a}{EQ} = \cos.\ AE$$

$$= \cos.\ x \quad \therefore \frac{b}{\cos.\ (x-a)} = \frac{a}{\cos.\ x} \qquad \text{or } \frac{b}{a} = \frac{\cos.\ (x-a)}{\cos.\ x} =$$

$$\frac{\cos.\ x\ .\ \cos.\ a + \sin.\ x\ .\ \sin.\ a}{\cos.\ x} = \cos.\ a + \tan.\ x\ .\ \sin.\ a$$

$$\therefore \frac{\frac{b}{a} - \cos.\ a}{\sin.\ a} = \tan.\ x$$

Calculation.

$$b = 150,\ a = 100. \ \tan.\ x = \frac{1{\cdot}5 - \cos.\ 1° 40'}{\sin.\ 1° 40'}$$

$$\log.\ \cos.\ 1° 40' \ldots \bar{1}{\cdot}999816 \qquad \therefore \cos.\ 1° 40' \ldots {\cdot}9994$$

$$\tan.\ x = \frac{1{\cdot}5 - {\cdot}9994}{\sin.\ 1° 40'} = {\cdot}5006\ .\ \operatorname{cosec.}\ 1° 40'$$

∴ x or lat. A=	86°	40′	30″	N
difference of lat.=	1	40	0	S.
lat. of B required=	85	0	30	N.

Prob. 183 (Fig. 106. Add to fig. the lines OX and PS).

Let XZ be the apparent zenith distance of sun, SZ the true zenith distance of sun, O the object whose bearing is required.

(1.) First suppose O to be the horizon.

In quadrantal triangle ZOX there are given, OX the observed distance of object from sun, ZX the observed zenith distance, and OZ=90°, to find the angle OZX.

(2.) In triangle PZS are given, PZ the co-lat., PS the polar distance of sun, and ZS the true zenith dist., to find the true bearing of sun, or the angle PZX.

The sum or difference of the two angles thus found, according to the position of the object with respect to the sun, will be the bearing PZO of object required.

Calculation.

(1.) To find angle OZX.

app.alt.sun's l. l. . . .	10°	30′	0″
index corr.	0	0	50−
	10	29	10
semi.	0	15	45
	10	44	55
dip.	0	3	41
app.alt.sun's center .	10	41	14
cor.for refraction,&c.	0	4	50
true alt. sun's center	10	36	24

ob. dist. n. l.	95°	16′	0″
in cor.	0	1	10+
	95	17	10
semi.	0	15	45
OX.	95	32	55

In quadrantal triangle ZXO,

cos. OX=sin. ZX . cos. OZX

or − + −

cos. OX . sec.app.alt.=cos.OZX

cos. OX	8·985491
sec. app. alt.	0·007600
180° 0′	
84 21	8·993091
OZX= 95 39	

(2.) To find PZS.

lat............	7° 51′ 0″			0·004089
tr. alt.........	10 36 24			0·007487
	2 45 24			4·926411
pol. dist......	112 24 0			4·912410
	115 9 24			9·850397
	109 38 36		∴ PZS	114° 39′
			and OZX	95 39
		bearing of point ..	N.	19 0

(2.) In the preceding example suppose O to be th
mit of a mountain whose elevation is 10°; it is requi
find the true bearing of the point O. In the precedi
culation, to find angle OZX we must substitute the
distance (80°) of the point O for the quadrantal arc
last example. We have then three sides given of the
OZX, to find as before the angle OZX.

Calculation.

ZO ..	80° 0′ 0″		0·006649
ZX ..	79 18 46		0·007594
	0 41 14		4·871860
OX ..	95 32 55		4 867153
	96 14 9		9·753256
	94 51 41	OZX ...	97° 39′
		and PZS	114 39
		bearing of mountain...N.	17 0 W

THE END.

LONDON:
ROBSON AND SONS, PRINTERS, PANCRAS ROAD, N.W.